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Mathematics as a creative art[1]

JULIA WELLS BOWER *of Connecticut College, New London, Connecticut,*
uses the works of Euclid to make her point,
namely that mathematics is a creative art
and that "...all of us...are ourselves artists of mathematics."

THE HUMAN consciousness appears to be a duality. On the one hand it is ever creating, and on the other hand it is ever holding fast to that which it has created. These two activities belong, respectively, in the realm of art and in the realm of knowledge. For the purpose of discussion, let us define art as the emergence of new life-forms of the human consciousness, and knowledge as the more or less permanent system of invariants which the human consciousness retains. [3:31][2]

Knowledge, then, can be considered, studied, and restated. Routes from one item of knowledge to another can be created and then held relatively stable. We call these routes logical reasoning. Each is based on a system of logic. One such system has been in continuous use for many centuries. The finding of a different route would be an act of creation, therefore an art. New systems are evolving, and we may soon find that our habitual mode of thinking is not the only one which carries us successfully from one item of knowledge to another.

Art, on the other hand, can not be restated. It is creative and, therefore, impermanent. It is characterized by the unexpected and the versatile. Even in the creative process, however, there are permanent realities. For instance, that which produces symmetry is a permanent reality, but its mode of expression is continually changing, producing new art. It is expressed in many ways in many fields. Consider some examples that are purely mathematical. In algebra the symmetries of the coefficients in the binomial expansion, of positive and negative, of the roots of unity; in geometry the symmetry in equations and properties of the central conics, and in the interchange of line and point in the Plücker equations. These examples have been very specific. Consider also the many evidences of symmetry revealed in mathematical literature. These occur in the comparison and contrast of theorem and converse, in the construction of many proofs, and even in the notation which mathematicians have invented to convey their meanings. Although these symmetries have long belonged to the realm of mathematical knowledge, still the discovery, investigation, and elucidation of each of them was an act of creation, therefore an art.

Once the human consciousness has observed a sufficient number of examples of the expression of any such reality as that which produces symmetry, then the stable idea of that reality becomes a part of knowledge. For instance, all of us know that symmetry exists. We seek for it and are delighted when we find it. Thus it is that in the duality of the human consciousness, the stable idea of reality is knowledge, the analysis of the reality in some form is art. Mathematics is, therefore,

[1] Paper presented at the meeting of the Association of Teachers of Mathematics in New England. December 6, 1952.

[2] The first numbers in square brackets refer to the list of references at the end of the paper. The second numbers give pages in the references.

both knowledge and art. As knowledge it possesses a large body of relatively permanent forms. These are its facts, its methods, and whole systems of thought. As an art, it reveals aspects of reality.

Let us pursue this last point a little farther. The presence within the body of mathematical knowledge of contrasting systems such as the Euclidean and the non-Euclidean geometries make it probable that mathematics is not limited to the discovery of some pre-ordained Truth, but rather is truly an art free to create whatever world of the imagination it chooses. The reality which we have been considering, then, is a reality only in the sense that it reveals itself to many men in many ways, and that men recognize a common element in these revelations. Differently constructed creatures on some different planet might create worlds of imagination quite different from ours. Then the realities whose various aspects would be revealed to them by common elements in these different worlds of imagination would, in turn, be quite different from ours. It follows that mathematics is a creative art, limited only by the potentialities of the men who create it, and revealing various aspects of what we conceive as reality.

Now if mathematics is a creative art, its works must range from the trivial received from the hands of a shallow artist to the important received from the hands of a master craftsman. We would do well to inquire what are the characteristics of good mathematics. These are hard to set down and sometimes appear to be contradictory. Let us consider synthetic geometry in the hands of Euclid. This is widely accounted good mathematics.

First of all, the concepts considered are general rather than specific and could apply to a variety of mathematical entities. Consider Euclid's first definition: "A point is that which has no part." [2:153] Notice how carefully the author divorces the mathematical concept of a point from any physical concept such as, "That which may be indicated by a mark." We are free to consider any mathematical construct satisfying the definition as a point. Some of the constructs with which we are familiar are: a pair of real numbers (x,y) and a complex number $x+yi$. There are less familiar ones. For instance, an infinite sequence of numbers instead of our accustomed two, x and y; or a polynomial of form ax^2+bx+c where a, b, and c are real. To be sure, Euclid himself probably never thought of such generalizations, but the possibility of such extension is present in his work.

A second characteristic is that the results obtained are general rather than specific. Take Proposition 20 of Book I. "In any triangle, two sides taken together in any manner are greater than the remaining one." [2:286] Or, in modern American, "In any triangle, the sum of two sides is greater than the third side." This theorem applies to triangles regardless of area, length of sides, relative sizes of angles, etc. It is not surprising, then, that we find it applied repeatedly, even appearing some two thousand years later in finding the analytic equation of an ellipse. This applicability is a third characteristic of good mathematics.

A fourth characteristic of Euclid's work has already been implied; namely, its fertility and suggestiveness for later mathematical constructions. The theorem quoted above suggests strongly an important concept of distance. In modern mathematics with its sophisticated generalizations we find that a distance function is defined for any set of points or elements by the properties that with each pair of elements, P_1 and P_2, there is associated a non-negative real number, $r(P_1, P_2)$, such that

$$r(P_1, P_2)=0$$

if and only if $P_1=P_2$,

$$r(P_1, P_2)=r(P_2, P_1),$$

and for any triple of elements, P_1, P_2, and P_3,

$$r(P_1, P_2)+r(P_2, P_3)\geqq r(P_1, P_3).$$

You recognize this third condition. It is even called the triangle inequality.

Now this distance function can have to do with ordinary lengths of ordinary straight lines if we define our points as pairs of real numbers (x, y) and let $r(P_1, P_2)$ be the familiar

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

On the other hand, if we let our points be infinite sequences of real numbers having values between 0 and 1 so that

$$P_1 \text{ is the point } (x_1, x_2, x_3, \cdots)$$

and

$$P_2 \text{ is the point } (y_1, y_2, y_3, \cdots)$$

where the x's and y's all have values between 0 and 1, then a satisfactory distance function $r(P_1,P_2)$ is given [4:6] by the infinite series

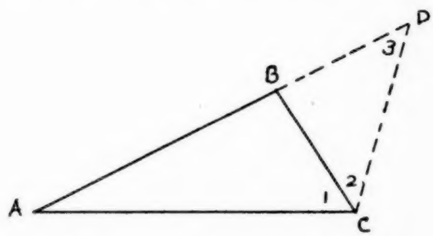
$$r(P_1, P_2) = \frac{|x_1-y_1|}{2} + \frac{|x_2-y_2|}{2^2} + \frac{|x_3-y_3|}{2^3} + \cdots.$$

Here the idea of distance is completely free from the idea of lines along which it might be measured. Our modern distance function is surely a concept which would have left Euclid himself bewildered, yet delighted with its generality.

As a fifth characteristic, Euclid's work on geometry contains methods of proof which are of wide applicability. The *Reductio ad Absurdum* which he so often employed comes immediately to mind. We will, however, discuss his direct proof. Underlying all his work is the insistence of the ancient Greeks that any proof shall consist of two parts, the *analysis* and the *synthesis*. [2:137] Because most of the propositions are relatively simple, Euclid generally left the analysis to the reader. Only in the case of certain theorems on proportion is the analysis included as part of the published proof.

What, then, is the analysis? The word *analysis* comes from two Greek words meaning *to loose back*. We use it now chiefly in the sense of carrying back to first principles. As used technically with reference to a proof, however, it meant a solution or argument backwards. According to Euclid, "Analysis is an assumption of that which is sought as if it were something admitted, (and the passage) through its consequences to something admitted (to be) true." [2:138] Its purpose is to suggest a method of proof for the theorem by a reversal of some or all of the steps. Let us make the analysis of Proposition 20 which says, you remember, that in any triangle the sum of two sides is greater than the third side. Since Euclid did not write out this analysis, we must reconstruct it according to the rules laid down by Pappus.

First we assume that our conclusion is true: that is, that the sum of two sides of a triangle is greater than the third side. To keep us all thinking of the same thing, let us draw and letter a triangle ABC, and let us agree that we will work particularly from the assumption that $AB+BC>AC$.



We remember that for Euclid, the sum of two sides would immediately suggest the construction of a line segment equal in length to the given sum. He would probably extend AB to D so that $BD = BC$. This construction is possible by use of Postulate 2: "To produce a finite straight line continuously in a straight line" [2:154] and Proposition 2: "To place at a given point (as at an extremity) a straight line equal to a given straight line." [2:244]

Now we consider all the propositions we know which start with the hypothesis

that two lines are unequal in length. There is only one in our experience so far. It is Proposition 18: "In any triangle, the greater side subtends the greater angle." [2:283] To use this proposition, we need a triangle having the two unequal line segments as sides. The use of ADC suggests itself. So we join D and C. This is possible by Postulate 1: "To draw a straight line from any point to any point." [2:154] Now we know that angle ACD opposite greater side AD is greater than angle ADC opposite the smaller side AC.

An inspection of the figure shows that angle ACD is made up of the sum of two angles, $\measuredangle 1$ and $\measuredangle 2$. Do we know anything about either of these? Yes, the two angles, $\measuredangle 2$ and $\measuredangle 3$, are base angles of the isosceles triangle CBD; and Proposition 5 says, in part, "In isosceles triangles, the angles at the base are equal to one another." [2:251] Angle ADC is therefore equal to a part of angle ACD and the inequality of these two angles implies that the whole angle ACD is greater than its part BCD. This is a restatement of Common Notion 5: "The whole is greater than the part." [2:155] This Common Notion 5 is surely the "something admitted to be true" which we have been seeking. We have, therefore, completed the analysis.

We must now determine whether or not we can infer, in the reverse order, that the conclusion of our original theorem is true. It is probable that we can, for the main proposition which we used, Proposition 18, has a converse which has already been proved. We start with the triangle ABC and make the construction, which we know is permissible, of extending side AB a distance BC to D and then joining D and C. We note the resulting isosceles triangle CBD with equal base angles, $\measuredangle 2$ and $\measuredangle 3$. Angle ACD is greater than its part, angle 2, therefore greater than angle 3 or angle ADC. It now follows that side AD opposite angle ACD is greater than side AC opposite angle ADC. This is true by Proposition 19: "In any triangle the greater angle is subtended by the greater side." [2:284] Since $AD=AB+BC$ we have reached the desired conclusion.

We have therefore put together in the proper order a proof beginning with the hypothesis and ending with the conclusion. The Greek word for "putting together" is *synthesis*. As Euclid defines it, "Synthesis is an assumption of that which is admitted (and the passage) through its consequences to the finishing or attainment of what is sought." [2:138] In other words, synthesis assumes the truth of the hypothesis and works from that to the conclusion.

You recognize in the analysis and the synthesis the twin elements in the method of successful attack on most mathematical problems, not only in geometry, but in any field of mathematics. Do these not form the orderly procedure which we try to follow? Frequently we do not carry the analysis all the way to the enunciation of the "something admitted to be true." We carry it only far enough to suggest the course of reasoning in the synthesis. But the analysis by which the proof is thought out is a necessary process. Do not most of our mathematical difficulties come from lack of attention to the analysis?

Next comes the task of writing up the synthesis so that it may be clearly presented to any reader. Euclid scrupulously follows the six steps prescribed by ancient Greek custom for properly constructing a proof. [2:129] First, the statement of the complete theorem. Second, the "setting out," that is, the telling of what is given adapted for use in the proof. In this case, "Let ABC be a triangle." Third, the "definition" giving a clear statement of what is sought: "$AB+BC>AC$." Fourth, the construction. Fifth, the proof itself which draws the required inference by correct reasoning from acknowledged facts. Sixth, the conclusion which states that what was to have been proved has been proved.

In Euclid's presentation of the synthesis, you recognize the form which is basic

to clear mathematical exposition. No matter how much experience a mathematician has had in mathematical writing, he can not deviate far from the orderly clarity imposed by this form and still have his writing understandable. Surely Euclid had the advantage and honor of being one of the first to exhibit methods of proof and exposition that are widely applicable.

From our discussion of Euclid's work, then, we conclude that to be good art, mathematical creations must have generality in concept which permits of wide applicability, must provide results which are general and of use, must be fertile in suggestions for later mathematical work, and may provide useful methods of proof. It is clear that if we should similarly consider the work of any other master, new criteria might emerge.

This eulogy of Euclid may have given the impression that his work on geometry is a perfect work of art. Not at all. It was constructed by a mortal man, and therefore was subject to error. The questioning of it began almost before the ink was dry on the manuscript. In modern times, a thorough critical study of Euclid's work showed defects. The structure has, consequently, been renovated with the result that it is more imposing than ever.

We turn now from the masterpiece to the artist, and ask how is mathematics created. How does the act of imagination occur which brings into being a work of mathematical art? On this point we have the testimony of a number of masters of the craft, compiled and analyzed by Jacques Hadamard. [1]

Hadamard believes that there is no single mathematical gift, that the mathematical faculty is composite. Further, mathematical creation and general intelligence are connected. Now, given an individual who possesses high mathematical aptitude, how does he go about creating a mathematical work of art?

According to Hadamard, he must first consider deeply the subject he has in hand. He must be working on a problem that has challenged him, using every effort to solve it by reviewing earlier related material, trying out methods which might apply to it, and bringing all pertinent knowledge which he has to bear on it. In other words, he must be immersed in his problem. If, while he is working, the solution of the problem suddenly appears to him, his act of mathematical creation has been achieved. If it does not come to him in a reasonable time, he will probably drop that problem for a while. Then, later on, in a sudden illumination, the solution may appear. Hence, at the time it appears, he may be working on the same or related problem, or a different one; or he may not be working at all. The illumination, when it comes, seems to take most mathematicians by surprise.

There are, then, three stages of mathematical creation: the stage of preparation, the stage of incubation, and the stage of illumination. Hadamard asks what is going on in the individual's mind during these three stages. During preparation, while the individual is immersing himself in the problem, many ideas are set in motion in his mind. During incubation, these ideas assemble and reassemble themselves into patterns in the unconscious. Most of these patterns are sterile and never reach the conscious, for the unconscious permits to enter into the conscious only those which are fruitful. Then at the moment of illumination, the conscious mind grasps the value of what has been laid before it and chooses the correct idea.

How does the unconscious know what ideas may be fruitful, and how does the conscious recognize the correct idea when it is laid before it? Certainly, mathematical creation requires discernment, choice. This choice may be governed by our aesthetic sense. For it is true that most mathematicians are profoundly moved by perception of reality such as we discussed in the early part of this paper. Harmony, symmetry, rhythm, elegance, many others, all of these may play their part. Note that we have come back again to

the idea that mathematics is a creative art, dependent on the imagination, perception, and discrimination in the mind of the mathematician creating it.

Hadamard suggests that this process of preparation, incubation, and illumination is exactly the one which goes on at any stage of mathematical learning. The difference between the work of Euclid and the work of a student trying to master one of Euclid's theorems is a difference of level. Both arrive at the proof of the theorem by the same process. Euclid was hampered by having no direction indicated in which he should proceed. He had to work out the analysis for himself and then write up his synthesis. The student, on the other hand, is hampered by having presented to him only the synthesis. He must work out from the synthetic presentation the underlying analysis. When he has done this and when his mind has grasped the analysis, then he understands the theorem. In the act of understanding, he has created new mathematics for himself.

Thus is it that all of us who are truly mathematicians, even though only learners at the feet of the great masters, share their spark of creativity and are ourselves artists of mathematics. In our own experience, then, we attest the fact that mathematics is a creative art.

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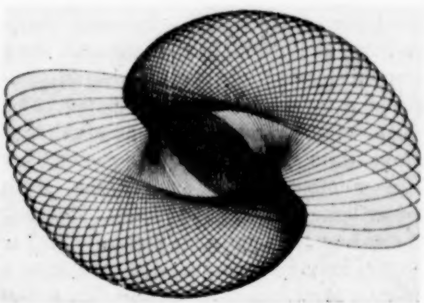
Pendulum patterns

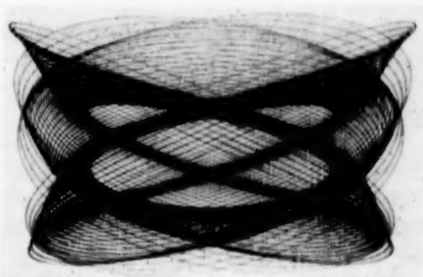
The curves at the bottom of this page were supplied to THE MATHEMATICS TEACHER by George T. Hillman, 923 Belle Plaine Avenue, Park Ridge, Illinois. These curves record the movement of a simple pendulum.

To produce these curves a small source of light is suspended on a cord and a camera, placed on the floor, is focused on the light source. The light source is then set in motion and the shutter of the camera opened for a time exposure.

The familiar Bowditch or Lissajous curves are special members of the group of curves produced by the pendulum method briefly described above.[1] The Bowditch curves are defined by the equation $x = a \sin(n + c)$, $y = b \sin t$ and were named after Nathaniel Bowditch, the well-known navigator who made a special study of these curves. They were first encountered in the study of the apparent motion of the earth from the moon. They later arose in the study of acoustics and the geodesic lines on certain surfaces.

[1] For a high-school student's work with pendulum patterns see THE MATHEMATICS TEACHER, Vol. XLV, p. 586.





Constructing graphs with the slide rule

CARL N. SHUSTER, *Professor of Mathematics at State Teachers College, Trenton, New Jersey, and past-president of the National Council of Teachers of Mathematics is convinced that high-school students should be introduced to semi-logarithmic graphs. His suggestions will prove to be of interest to many mathematics teachers.*

THE SLIDE RULE is an invaluable aid in the construction of certain types of graphs. The familiar circle graph is the best for showing the relationship between the various parts of a whole and the whole. It requires a considerable amount of work to find the central angles for the various sectors. All this work may be done on the slide rule in a fraction of a minute.

Suppose a family of four spends an income of \$4250 as follows:

Item	*Amount*	*Angle*	*Angle Rounded*
Food	\$1360	115.2°	115°
Shelter	1190	100.8°	101°
Clothing	595	50.4°	50°
Savings	510	43.2°	43°
Recreation	340	28.8°	29°
Health	255	21.6°	22°
Total	\$4250	360.0°	360°

To find the angles (given in the third and fourth columns) set 4250 (C scale) over 360 (D scale). Then find the amounts 1360, etc., on the C scale and read the angles 115.2, etc., directly under on the D scale. The angles as read from a 20-inch slide rule check perfectly but are too accurate for the graph and are rounded in the fourth column.

Another form of graph used to show the relationship between the parts and the whole is the divided-bar graph. Such graphs are often constructed on a sheet of plain drawing paper or poster paper.

To construct a divided-bar graph to show the same facts shown in the circle graph discussed above, measure the length you wish the bar to be. Suppose this is 38 centimeters. Set 4250 on the C scale over 38 on the D scale. Under 1360, 1190, etc., read 12.16, 10.64, 5.32, 4.56, 3.04, and 2.28. These are the sections of the bar. The total of these is 38.00. These lengths are too accurate for the purpose and should be rounded to 122 mm., 106 mm., 53 mm., 46 mm., 30 mm. and 23 mm. Divided-bar graphs may be made very quickly by cutting off the required lengths of colored scotch tape and placing these sections on the bar.

There are many places in business, industry, and engineering where the relative change, relative gain, relative loss, relative error, etc., is far more important than the actual change, gain, loss, or error. An increase of \$3 a share on \$12 stock is far more significant than an increase of \$3 a share on a \$120 stock. An error of 0.4 feet is far more serious on a 3.0-foot measurement than an error of 0.4 feet on a measurement of 12,568.0 feet. In all such cases graphs showing change should be constructed on semi-logarithmic paper. Also when it is necessary to plot numbers that range from 1 to 10,000, or some other large number, on the same graph, semi-logarithmic paper is necessary. Figure 2 shows how the number of counts from 2 to 10,000 may be recorded and read from a 10-inch semi-logarithmic chart. To record

Figure 1

Stocks are usually graphed on semi-logarithmic paper.

and read the same data on an ordinary chart, the chart would have to be 156 inches long. On a semi-logarithmic chart the logarithms of numbers are plotted rather than the numbers. Since the A, B, C, D, and K scales on a slide rule are scales of logarithms of numbers, they are graduated exactly like the ordinates of semi-logarithmic paper.

To see clearly the difference between ordinary graphs and semi-logarithmic graphs let us assume that a 12-lb. baby, a 120-lb. woman, a 1200-lb. horse and a 12,000-lb. elephant have the same actual gains and losses in a period of six weeks as follows: +2 lb., +3 lb., +1 lb., −2 lb., −3 lb., −4 lb. Plotted to the same scale on the same sheet of ordinary graph paper, all four graphs would have exactly the same shape but would differ in location. The sheet would have to be at least 12,000 units high or if each unit was 0.1 inch it would be 1200 inches or 100 feet high. Plotted on polyphase semi-logarithmic graph paper 18 inches high, all the graphs would show clearly. The graph for the elephant would be practically a straight line and that for the horse would be quite smooth. The graph for the woman would show more fluctuation but would not indicate a serious condition if we assumed she went on a vacation for three weeks and then on coming home went on a diet for three more. If, however, the downward trend continued for several weeks more she should most certainly see a doctor. The graph for the child would indicate a very serious, perhaps fatal, situation. The semi-logarithmic graphs show the relative change rather than the actual change. These graphs are widely used today and their use is increasing so rapidly that they can no longer be ignored by the schools.

Semi-logarithmic paper is expensive and not always at hand. It is hard to secure the exact size needed. However, an ordinary slide rule gives a considerable selection of semi-logarithmic scales. To use the slide rule for this purpose construct lines perpendicular to the base line as far apart as is desired. Then to locate numbers on these lines place one of the scales of the slide rule alongside and mark off the log of the number. If one-phase paper is wanted (1–10), the C scale may be used. If two-phase paper is desired (1–100), the B scale should be used. If three-phase paper is desired (1–1000), the K scale should be used. Two of the three parts of the K scale of a 10-inch rule may be used to get two-phase paper about 6 inches high. Four of these sections may be used to get four-phase paper about 12 inches high. If the user has a 5-inch rule, a 10-inch rule and a 20-inch rule, he can produce any phase semi-logarithmic paper of almost any desired size.

The engineer or scientist needing log-log paper can readily construct it by using the log scales for both abscissas and ordinates.

If a teacher wishes to have the class construct slide rules, this may be done by buying one-phase and two-phase semi-logarithmic charts of the same length, cutting them in strips and using them for C–D and A–B scales. If a K scale is desired, a sheet of three-phase paper of the same length may be used. Quite a few strips may be cut from one sheet. If the strips of semi-loga-

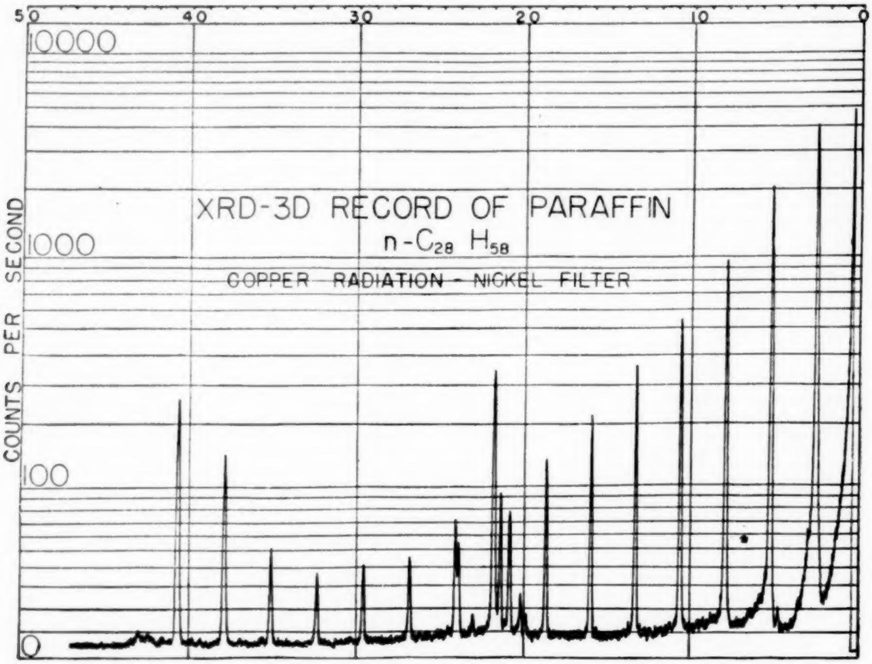


Figure 2

It would take a chart 13 feet high to give you this range of reading on a linear scale. The General Electric XRD-3 logarithmic scale permits you to read from 2 to 10,000 counts per second on a single 10-inch scale.[1]

rithmic paper are pasted onto wood forms and then varnished and waxed they will last several years.

This paper is written for teachers with a fair knowledge of the slide rule. It is assumed that they will be able to show their students the fundamental mechanics of the slide rule. Meaning is just as important in slide rule computation as in any other phase of mathematics.

[1] Used by permission of General Electric Company.

Summary of research

The United States Office of Education has made available a "Summary of Research Studies in Mathematics Education." This summary is the culmination of a project jointly sponsored by the United States Office of Education and The National Council of Teachers of Mathematics.

Report forms were mailed to leaders in research in mathematics education early in 1953 requesting data on the research completed during the calendar year 1952. Fifty-seven studies were reported and the major findings summarized.

Teachers of mathematics will be interested in seeing what problems are of interest to those who were doing the research in the field of mathematics education for the year 1952.

Copies of this circular may be secured from the Publications Section, Office of Education, U.S. Department of Health, Education, and Welfare, Washington 25, D.C.

Imaginaries

FRED GRUENBERGER *of the Numerical Analysis Laboratory, University of Wisconsin, calls attention to a "non-imaginary" application of the imaginary numbers which, unless properly presented, prove to be really imaginary for the pupil.*

"IMAGINARY" numbers are usually presented in algebra classes merely as a necessary (and awkward) evil arising from the solution of quadratic equations. Indeed, one may go quite far in mathematics before discovering some of the ingenious practical uses of complex numbers.

If we number the axes of a Cartesian coordinate system with the usual real numbers on the x-axis, and imaginary numbers on the y-axis, we form the complex plane. In the complex plane, algebraic addition of complex numbers plots as vector addition. Thus the sum of two complex numbers has as its geometric counterpart the resultant of the parallelogram of forces.

This gives us a powerful tool for handling problems in alternating-current theory. Many high-school students are familiar with enough direct-current theory to know that the resultant of two pure resistances in parallel is given by

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$$

or

$$R=\frac{R_1R_2}{R_1+R_2}\,.$$

In the field of alternating current an electrical element seldom exhibits pure resistance. A coil has pure resistance, as measured by direct current instruments, plus a resistance to the alternating current known as inductive reactance (measured in ohms by the product of 2π times the frequency times the inductance of the coil, measured in Henries). The net opposition the coil offers to an alternating current is the resultant of the two types of resistance and is called the impedance. Impedance is also measured in ohms, and is usually denoted by the letter Z.

The impedance is the vector sum of the pure resistance and the reactance; that is, its magnitude is the diagonal of the rectangle of which the resistance and reactance are the sides. Thus if we plot the resistance on the x-axis and the reactance on the y-axis, then the point so located is the impedance, the distance from the origin being the magnitude.

For example, if a coil has 3 ohms of pure resistance and 4 ohms of reactance (due to a particular frequency), then at that frequency its impedance is 5 ohms. As a matter of standard practice, inductive reactance is plotted in the direction of the positive y-axis. The resulting impedance is represented by the complex number; in this case $3+4i$. Reactance and impedance are quite real quantities; plotting them on the complex plane and representing them by "imaginary" numbers is simply a device to perform the necessary vector algebra.

The argument of the complex number (arctangent 4/3 in the example) is a real quantity also. It is the phase angle by which such a coil induces the current to lag behind the voltage.

Another electrical element might have a pure resistance of 4 ohms, but oppose the

flow of alternating current to the extent of 3 ohms due to condenser action. This is capacitative reactance, and is measured by the reciprocal of the product of twice pi, the frequency and the capacitance.

The impedance may be found by plotting on the complex plane as before, except that in the case of condensers the reactance is plotted in the negative direction on the y-axis. The impedance in this case would be $4-3i$, and the argument of the complex number indicates that the current leads its voltage by the angle arctangent 3/4.

Now, it can be shown that the law given previously for resistances in parallel holds true for impedances in parallel, so that

$$Z=\frac{Z_1Z_2}{Z_1+Z_2}\,.$$

If the two elements we have considered are mounted in parallel, the resulting impedance is given by the formula for Z. The resultant, $3.5+.5i$, is a very real quantity (about 3.535 ohms, 8.2 degrees inductive) although we used complex numbers to find it. Incidentally, electrical engineers prefer the letter j to i, and refer to this tool as the "j operator."

Have you read?

Carnap, Rudolf. What Is Probability? *Scientific American*, September 1953, p. 128.

Is probability "statistical" or "inductive"? Is it an established fact or a means of measurement based on available evidence? This question has plagued the users of probability down through its entire history from Bernouille to Laplace to R. A. Fisher. The pendulum has swung in both directions. With the rapid increase in the use of statistics the author shows the need for both interpretations, but the more serious need is for probability as an inductive process. "Ars Conjectandi," the art of conjecture, is needed more today than ever before. The theory of probability can supply a means of evaluating the acceptability of assumptions or hypotheses. This article places probability in its rightful place as an aid in making judgments.

Love, Malcolm A. "Accent on Responsibility: The Responsibility of the Educated, California Schools, August 1953, p. 360.

You will want to read President Love's address on this question which is so vital to the future of our country. He explains why education must be considered a method of increasing one's power to accept greater responsibility. Can you be free and disciplined simultaneously? How have we confused activities which make for comfort as synonymous with those which make for greatness? Are we a great nation because of vast resources and national wealth or because our society permits the unlimited growth of the person, and assumes his ability to make decisions according to a plan developed by himself? All this and more the educator must strive to meet. He must be humble, but he must hold opinions while at the same time taking the responsibility of identifying them as opinions. This article presents a serious problem to all of us in the educational profession.

Wedul, Melvin O. "Grains of Sand and Drops of Water Help Make Numbers Meaningful," *School Science and Mathematics*, April 1953, p. 294.

A light year is a great distance but do you know how far? Can such an enormous quantity have meaning to the earth-bound creature six feet tall? You will want to read what Mr. Wedul did with drops of water and grains of sand to make 12 to 14 digit numbers meaningful. It is still true, as Julia Carney stated long ago, that

"Little drops of water and little grains of sand
Make the mighty ocean and the pleasant land."

"What It Costs to Run a Car," Changing Times, *The Kiplinger Magazine*, July 1953, p. 13.

The teacher of arithmetic will find in this article a great deal of material that will aid him in making arithmetic meaningful and developing in the pupil a valid concept of quantity. Some questions discussed and interpreted are as follows: What affects depreciation? How much is depreciation per year? What are all the fixed costs or operational costs? What incidentals might be expected? How does total mileage affect the cost? These questions and typical statistics about them will surprise you as well as the pupils. This is a good introduction to approximation or judgments.—Philip Peak, Indiana University, Bloomington, Indiana.

Direct vs. indirect memory in geometry

ROLAND B. KIMBALL *of Concord High School, Concord, New Hampshire, reports on the results of an informal experiment conducted at Concord High School. Roland Kimball's tentative conclusion that ". . . it seems reasonable to infer that those who do not memorize (geometry) tend to remember more effectively than those who do" should furnish food for thought for mathematics teachers.*

NOT TOO LONG ago it was common practice among geometry teachers to expect pupils to memorize much, if not all, of the work in plane geometry. Not only were the theorems memorized; so were the proofs. Fortunately, this sort of thing is on the decline. The trend is toward the use of frequent original proofs which can be done by the pupil only if he truly understands the theorems which apply.

But even with this approach the pupils are often expected to know (that is, to remember) the important propositions. When a test is given, the pupil is on his own. He cannot refer back to either textbook or class notes. In this way some type of formal memorization still is demanded of the student.

Now let us suppose a pupil were given a course in plane geometry which completely disregarded memorization. Suppose that during every test and quiz he were permitted to refer to any books or notes which he felt would help him to do a better job. Would he fare better under this plan or under the more conventional one? That is, at the end of the course, under which plan would he have learned the most geometry?

Of course, it is not possible to answer such questions with certainty. However, a brief study made by the writer does offer some interesting evidence on this point.

During the 1951–52 school year some of the geometry sections in our senior high school were taught by what we will call the "open book" system while other sections were taught under a "closed book" method. All of the sections received exactly the same course in geometry (as closely as reasonable under two different instructors). The main difference was this: the "open book" group never had to memorize any part of the work. Pupils from this group were permitted free access to books and notes during all tests and class discussions. Any memorizing, if that is the word to use, was indirect, incidental.

On the other hand, those in the "closed book" sections were permitted no references of any sort during tests. Memorization of the important propositions of geometry was expected and required.

In order to compare the "direct" and "indirect" memory groups, pupils from the two groups were paired. This pairing was done on the basis of (1) intelligence as indicated by the pupils' I.Q. scores; (2) grade in first year algebra; (3) score made on the Orleans Geometry Prognosis Test administered during the first week of the geometry course; (4) sex; (5) high school class, i.e., sophomores paired with sophomores, juniors with juniors, etc. On this basis it was possible to select twenty-one reasonably well matched pairs of pupils. Table 1 summarizes the data concerning the two groups, each consisting of twelve boys and nine girls.

One might ask whether the differences

TABLE 1

	Orleans Geometry Prog. Test Score				Grade in First Year Algebra				I.Q.			
	Low	High	Mean	S. D.	Low	High	Mean	S. D.	Low	High	Mean	S. D.
Direct memory group	66	151	110.8	21.9	74	90	82.9	5.5	93	129	110.5	8.6
Indirect memory group	68	144	110.8	20.9	75	95	86.1	5.5	94	125	109.5	7.3

between the two groups are large enough to be significant. Obviously, unless the groups are similar, any later comparison of their performances would be meaningless. Just from inspection of the data contained in Table 1, one would be inclined to say the groups are reasonably identical.

However, there are statistical devices which may be used to test the significance of the differences between the standard deviations of small sample groups (Snedecor's F-test) and between the means of such groups (the t-test). For a discussion of these tests the reader is referred to any standard text on educational statistics. These tests were applied to the data in Table 1 and indicated that the two groups were comparable on the basis of the three measures used for pairing.

At the end of the school year the pupils were given the Cooperative Plane Geometry Test, Revised Form, Series T. This is a conventional test of geometry achievement with emphasis on recall and comprehension.

This particular test was "closed book" for all pupils. The "open book" group knew of this test about two weeks before it was given. But it is the opinion of the writer that none of the pupils resorted to any intensive, or even superficial, memorizing at the last minute. Certainly they were not encouraged to do so.

Table 2 summarizes the performances of the two groups on this achievement test.

The scores reported are the "scaled scores" reported on all the tests of this particular series. It will be noted that there is a difference between the averages of 6.3 points in favor of the indirect memory group.

TABLE 2.—Scores Made on the Cooperative Plane Geometry Test

	Low	High	Mean	S. D.
Direct memory group	42	66	53.0	6.8
Indirect memory group	48	67	59.3	6.5

Is this difference large enough to prove anything? That is, if we were to assume there is no particular difference in the geometry achievement of pupils taught under the two systems, then what is the probability that the difference between the group averages could be as large as 6.3 from chance alone? Once again, the most objective answer can be obtained by the use of the statistical devices mentioned earlier.

The F-test was applied and it indicated that the standard deviations of the two groups do not differ significantly. Thus, it is justifiable to use the t-test to see whether the difference between the means is a significant one. (The t-test should be applied only when the groups being compared are drawn from samples having approximately the same standard deviation.)

Computation of t yielded a value of 3.285. In our particular example such a t-value may be interpreted as meaning: the chances are fewer than one in a hun-

dred that t could be this large with no real difference in the averages of the two groups.

Stated another way, our inference is that the group which was not required to do any memorizing actually did a superior job on the geometry achievement test. Further, it seems reasonable that this superiority is not just a chance occurrence.

Now this achievement test, like so many other similar tests, requires that the pupil recall considerable factual information. Thus it seems reasonable to infer that those who do not memorize tend to remember more effectively than those who do. Teachers of geometry would do well to give this point some thoughtful attention.

In closing, it hardly seems necessary to observe that this study is not a rigorous scientific experiment. In several respects it is quite inconclusive. (1) The difference in achievement between the two groups may have been caused by other factors. We have no way of being certain the differences were due to the method of indirect memory as contrasted with direct memory, although this *was* the major variable which distinguished between the two groups. (2) We have no information at this writing as to which group will remember and/or achieve best after a period of time—say a year or more. Possibly the picture will be reversed. (3) All that has been tested is achievement in traditional geometry. We have no information as to which system, if either, best achieves that cherished objective of plane geometry—the capacity for logical thought in the situations of everyday life.

Bearing in mind these and perhaps other reservations, the evidence presented here still seems to indicate one tentative but very important conclusion. This conclusion is that the sort of memory which is built up through understanding and frequent, intelligent use of the geometric propositions is at least as potent as the traditional word-for-word memory.

Perhaps, then, here is one way to make high-school geometry more palatable without being accused of making it less productive.

Computers

The Burroughs Corporation has announced the new high-speed computer the Burroughs Laboratory Computer-Model I. This computer is largely made up of more than 750 standard electronic building blocks, known as Burroughs Pulse Control Units, which are widely used as test equipment in electronic research on computers. The time required for an addition in the computer is 17/1000 of a second (.017); for multiplication, 50/1000 of a second (.050).

Inquiries concerning the Burroughs Computation Service may be addressed to the Electronic Instruments Division, Burroughs Research, 511 N. Broad, Philadelphia 23, Pennsylvania.

"The recent analysis undertaken by Dr. Enid Charles shows that whatever changes in fertility and mortality may conceivably conspire to avert a rapid decline of net population from 1945 onwards, nothing can now forestall a rapid and spectacular depletion of the school age groups during the *next two decades*. Therefore the choice lies between a period of acute unemployment for teachers or a drastic reform of educational routine. No teacher should teach for more than ten hours a week. By 1950 an enormous reduction of working hours can be achieved without any increase in the cost of education."—L. Logben, *Mathematical Gazette*, XXII (1938), 122.

"The improvement of space perception ability is not being accomplished in the teaching of solid geometry."—This statement is based on the study by Ernest Raymond Ranucci. "Effect of the Study of Solid Geometry on Certain Aspects of Space Perception Abilities." Ph.D., 1952, Teachers College, Columbia University, New York City.—Major Faculty Adviser, Professor Howard F. Fehr.

Pythagorean numbers

PHILIP J. HART, *Associate Professor of Physics at the Utah State Agricultural College, Logan, Utah, presents an interesting tabular arrangement of the well-known Pythagorean numbers.*
Pythagorean sets have interested mathematicians for over 2000 years. They may be of interest to some students in the mathematics classes of the senior high school and the college.

Introduction and History

The "theorem of Pythagoras," to the effect that the square of the hypotenuse of a right triangle is equal to the sum of the squares of the two sides, is well known to every student of elementary geometry. No other geometrical theorem has been of greater importance throughout the history of mathematics and physics.[1] If x and y represent the lengths of the two sides and z the length of the hypotenuse of a right triangle, then in algebraic symbols the theorem states

$$x^2+y^2=z^2. \tag{1}$$

The primeval history of the Pythagorean theorem is lost in antiquity. Recently, Professor O. Neugebauer of Brown University has shown that a clay tablet from old Babylonian times, 1800–1600 B.C., contains a table of integers satisfying the Pythagorean equation.[2] Moreover, the table is not a random collection but is so constructed that the ratio of the hypotenuse to one of the sides decreases almost linearly through the fifteen sets of values in the table, and the values given for this ratio could serve with little error as a table of cosecants for degrees from 45° to 31°. The ancient Babylonians also evaluated $\sqrt{2}$ to an accuracy of better than one part in 600,000; thus, without doubt they possessed the "Pythagorean theorem" more than a thousand years before Pythagoras.

Although the ancient Egyptian "rope-stretchers" may have used ropes of lengths 3, 4, and 5 to construct right angles no evidence has been found that they knew the Pythagorean theorem,[1] or even that they recognized that $3^2+4^2=5^2$. In pre-Pythagorean Hindu writings the following number sets are given:[3] 3, 4, 5; 5, 12, 13; 7, 24, 25; and 15, 36, 39.

Pythagoras (about 569–500 B.C.) and his followers knew of the theorem which now bears his name, but whether from others or through rediscovery we do not know. The Pythagoreans may have given a proof of the theorem; in any event their most important contribution was the emphasis they placed on proof in mathematics.[4]

The oldest proof extant[5] for the Pythagorean theorem is that which Euclid gives in his Proposition 47, Book I. Numerous proofs have since been given and, in fact, are still being presented; a recent example of such a proof is given on page 473 of the October 1952 issue of The Mathematics

[1] E. T. Bell, *The Development of Mathematics* (2d ed.; New York: McGraw-Hill, 1945), pp. 40, 68.

[2] O. Neugebauer, *The Exact Sciences in Antiquity* (Princeton Univ. Pr., 1952), pp. 35–41.

[3] G. A. Miller, *Historical Introduction to Mathematical Literature* (New York: Macmillan, 1921), p. 158.

[4] E. T. Bell, *Men of Mathematics* (New York: Simon & Schuster, 1937), p. 21.

[5] F. Cajori, *A History of Mathematics* (2d ed.; New York: Macmillan, 1931), pp. 18, 56.

TEACHER, Vol. XLV. However, the concern of the present discussion is not with proofs of the theorem but rather with the "sets" of *integers* which satisfy equation (1). These integral solutions of the equation are called *Pythagorean numbers.*

Systematic attempts at finding Pythagorean numbers have been made by many men. Pythagoras is reputed by later writers to have given the equivalent of the following formulas:[5,6,7,8]

$$(2)\qquad \begin{aligned} x&=2n+1\\ y&=2n^2+2n\\ z&=2n^2+2n+1 \end{aligned}$$

where n is any integer. However, these formulas do not represent the general solution of equation (1) since they yield only the values found in the $a=1$ column of Table 1.

At a later date, Plato (430–349 B.C.) gave the following as Pythagorean number formulas:[6]

$$(3)\qquad \begin{aligned} x&=n^2-1\\ y&=2n\\ z&=n^2+1 \end{aligned}$$

For n even these formulas give the values in the $b=1$ row, and for n odd, twice the values in the $a=1$ column, Table 1.

It was Euclid (365?–275? B.C.) who first gave the *general* positive integer solution, in his Elements X, 28:

$$(4)\qquad \begin{aligned} x&=m^2-n^2\\ y&=2mn\\ z&=m^2+n^2 \end{aligned}$$

where both m and n are integers. This solution seems to be the first general integral solution of an indeterminate equation.[1] Neugebauer, however, is of the opinion that the ancient Babylonians also must have been in possession of this general solution.[2] Pythagoras' formula (2) can be obtained by setting $m=n+1$ in equations (4). Diophantus of Alexandria (c. 250 A.D.), the great contributor to algebra, was familiar with the general solution.[7]

In India in the seventh century A.D., Brahmagupta (c. 625 A.D.) gave two sets of formulas for Pythagorean numbers,[9] which, however, he probably obtained from Greek sources, including the general solution and a more complicated expression. Bhaskara (1114-c. 1185 A.D.) repeated Brahmagupta's formula and added two more.[9] However, these rather complicated special solutions are of no great concern or value to us except to show the continuing interest in Pythagorean numbers through the centuries.

It should be pointed out that the solutions obtained prior to the time of Diophantus were discovered without the use of modern methods of algebra and without our convenient Arabic numerals; thus the discovery was far more difficult than would be the case at the present time.

In recent times modern number theory has been used in different ways to prove that equations (4) contain the general primitive solution of equation (1). For instance, Klein[10] (1849–1925), using a geometric model in which he considers the conditions for which the fractions x/z and y/z are rational, and Ore,[11] using algebraic theorems and reasoning, both derive this solution.

THE TABULAR PRESENTATION OF PYTHAGOREAN NUMBERS

A general Pythagorean solution need give only positive integers having no com-

[6] J. Gow, *A Short History of Greek Mathematics* (Cambridge: Cambridge Univ. Pr., 1884), pp. 71, 155.

[7] W. W. R. Ball, *A Short Account of the History of Mathematics* (3rd ed.; London: Macmillan, 1901), p. 29.

[8] V. Sanford, *A Short History of Mathematics* (Boston: Houghton, Mifflin, 1930), p. 272.

[9] D. E. Smith, *History of Mathematics, Vol. I* (Boston: Ginn, 1923), pp. 160, 281.

[10] F. Klein, *Elementary Mathematics from an Advanced Standpoint* (Trans. from the 3rd German ed. by E. R. Hedrick and C. A. Noble (New York: Macmillan, 1923), pp. 44–46.

[11] O. Ore, *Number Theory and Its History* (New York: McGraw-Hill, 1948), pp. 167–170.

mon divisor. The generally accepted equations (4), however, will yield such solutions, first, *only if m is greater than n*, otherwise x is negative and duplication of absolute values occurs as well; second, *only if m and n are relatively prime*, since a common factor for m and n results in that factor squared for x, y, and z; and third, *only if m and n are not both odd*, otherwise x, y, and z are all even numbers and thus not relatively prime. The first and third facts apparently are not recognized by Klein in his derivation of the general solution.

A table containing only positive and primitive sets of Pythagorean numbers from equations (4), set up with values of m heading the columns and values of n the rows, has an excessive number of blank spaces, including the whole lower left half of the table where x is negative, and, in the remaining half of the table, the checkerboard of spaces for which m and n are both even or both odd, and, of course, all other places where m and n have a common divisor.

The serious shortcomings of the foregoing method of tabulation are evident, and so such tables as have been given in the past are presented in other ways. As an example, Martin[12] gives a "Table of Prime Rational Right-Angled Triangles Whose Hypotenuses Are Less Than 3000," which contains Pythagorean numbers listed in the order of increasing values of the hypotenuse.

In order to make Pythagorean numbers available in a useful tabular form, the author would now like to propose the following modified form of the general solution:

$$x = a^2 + 2ab$$
$$(5) \qquad y = 2b^2 + 2ab$$
$$z = a^2 + 2b^2 + 2ab$$

where a is an odd integer and b is any integer having no divisor common to a.

[12] A. Martin, "On Rational Right-Angled Triangles," *Proceedings of the Fifth International Congress of Mathematicians, Vol. II* (Cambridge University Press, 1913), pp. 40–58. Table pp. 57–58.

Table 1 is a presentation of some Pythagorean numbers according to this system.

Formulas (5) may be obtained from formulas (4) by the simple substitution of $m = a + b$ and $n = b$. This substitution insures that $m > n$ for positive a and b, thus eliminating negative values of x. The restriction of a to odd values is the equivalent in equations (4) of requiring that when m is even, n must be odd, and *vice versa*, since $a = m - n$. In numerical computation if the three numbers a^2, $2ab$, and $2b^2$ of equations (5) are first obtained, then x, y, and z are found simply by adding combinations of these numbers, the labor involved differing little from that for the somewhat simpler appearing equations (4).

Note that the expressions for x and y, both in equations (4) and in (5), are factorable. These factors, designated as s and r, are the basis of a general solution proposed on pages 269–70 of The Mathematics Teacher, Vol. XLV, April, 1952. In equation [3] as there presented, s and r are the factors of x, and in equation [4] s and r, properly taken, are the factors of y. Parenthetically, the restriction there given, that $s < r$, is not necessarily true for equation [4].

Derivation of Pythagorean Formulas

The main purpose of this discussion, that of proposing a convenient method for the tabulation of Pythagorean numbers, has now been accomplished. However, it may be instructive and of interest to show other methods by which equations (5) may be obtained.

Some time ago, the author, naively unaware at the time of most of the foregoing, sat down one afternoon to the little task of preparing a table of Pythagorean numbers which perhaps might prove useful in vector problems for elementary students, especially for use during examinations where the time for arithmetic calculations is short. He had observed some years previously that such numbers could be ob-

TABLE 1.—PYTHAGOREAN NUMBERS

The tabulation here presented is according to the system: $x=a^2+2ab$, $y=2b^2+2ab$, and $z=a^2+2b^2+2ab$, where a is an odd integer and b is any integer having no divisor common to a.

b \ a	1	3	5	7	9	11	13	15	17	19
1	3 4 5	15 8 17	35 12 37	63 16 65	99 20 101	143 24 145	195 28 197	255 32 257	323 36 325	399 40 401
2	5 12 13	21 20 29	45 28 53	77 36 85	117 44 125	165 52 173	221 60 229	285 68 293	357 76 365	437 84 445
3	7 24 25		55 48 73	91 60 109		187 84 205	247 96 265		391 120 409	475 132 493
4	9 40 41	33 56 65	65 72 97	105 88 137	153 104 185	209 120 241	273 136 305	345 152 377	425 168 457	513 184 545
5	11 60 61	39 80 89		119 120 169	171 140 221	231 160 281	299 180 349		459 220 509	551 240 601
6	13 84 85		85 132 157	133 156 205		253 204 325	325 228 397		493 276 565	589 300 661
7	15 112 113	51 140 149	95 168 193		207 224 305	275 252 373	351 280 449	435 308 533	727 336 625	627 364 725
8	17 144 145	57 176 185	105 208 233	161 240 289	225 272 353	297 304 425	377 336 505	465 368 593	561 400 689	665 432 793
9	19 180 181		115 252 277	175 288 337		319 360 481	403 396 565		595 468 757	703 504 865
10	21 220 221	69 260 269		189 340 389	261 380 461	341 420 541	429 460 629		629 540 829	741 580 941
11	23 264 265	75 308 317	135 352 377	203 396 445	279 440 521		455 528 697	555 572 797	663 616 905	779 660 1021
12	25 312 313		145 408 433	217 456 505		385 552 673	481 600 769		697 696 985	817 744 1105
13	27 364 365	87 416 425	155 468 493	231 520 569	315 572 653	407 624 745		615 728 953	731 780 1069	855 832 1193

tained from the equation $x^2+y^2=z^2$ by assuming that x, say, differs from z by an integer d; thus $z=x+d$. Substituting this value of z in the first equation yields $x=(y^2-d^2)/2d$, which gives a rational value of x for assumed integral values of y and d. It might have been observed that multiplying the x, y, and z values thus ob-

tained by $2d$ yields the solutions y^2-d^2, $2dy$, and y^2+d^2, the familiar equations (4) in different symbols; but if this observation *had* been made at this time it is likely that this article would never have been written. Instead, the observation was made that the values obtained for x, y, and z, when reduced to integers *having no common divisor* were always such that y and x differ from z by the square of an odd integer a or by twice the square of any integer b relatively prime with respect to a. Thus, if we let $x=z-2b^2$ and $y=z-a^2$, substitute into $x^2+y^2=z^2$, and solve for z by the quadratic formula, we obtain the value $z=a^2+2b^2\pm 2ab$; thus, also, $x=a^2\pm 2ab$ and $y=2b^2\pm 2ab$. If we discard the negative signs, which yield no new absolute values, this solution is the one given previously in equations (5) and represents the author's first method of obtaining it.

Equations (5) may be obtained in a less empirical manner from a general solution of equation (1), as follows:

Let the numbers by which x and y differ from z be d and c, respectively; then $x=z-d$ and $y=z-c$. Substituting these expressions into $x^2+y^2=z^2$ and solving for z by the quadratic formula yields $z=c+d\pm\sqrt{2cd}$. Thus

$$
\begin{aligned}
x &= c\pm\sqrt{2cd}\\
y &= d\pm\sqrt{2cd} \qquad (6)\\
z &= c+d\pm\sqrt{2cd}
\end{aligned}
$$

These equations represent solutions of $x^2+y^2=z^2$ for *any* assumed values of c and d, rational or irrational. It is evident that all integral values for c and d do not yield integral values for x, y, and z. However, if x, y, and z are to be integers, the differences c and d must be integers, and so the quantity $\sqrt{2cd}$ must also be an integer. Thus, the problem of finding Pythagorean formulas resolves itself into finding algebraic expressions for c and d which will make the expression $(2cd)$ a perfect square. An infinite number of such algebraic expressions is possible.

The most obvious substitution which will make $(2cd)$ a perfect square is to let, say, $c=a^2$ and $d=2b^2$. These values substituted into equations (6), by discarding the minus signs which lead to no new absolute values, yield the equations (5) as obtained previously. The commonly given equations (4) may be obtained by the less obvious substitution of $c=(m-n)^2$ and $d=2n^2$.

Special solutions are easily obtained, also, from equations (6). The substitution of $c=1$ and $d=2n^2$ results in equations (2), the substitution $c=(n-1)^2$ and $d=2$, results in equations (3), and so on. What would not those ancient peoples who placed so much importance on Pythagorean numbers and formulas have given for the methods of modern algebra!

More Dimensions

The problem of Pythagorean numbers is related to the broader problem known as Fermat's Last Theorem, a theorem which so many people during the last three centuries have vainly attempted to prove or disprove. This famous theorem, discovered about 1637, states that there are *no* integers which will satisfy the equation $x^n+y^n=z^n$, where n is any integer greater than 2.

It is well known, however, that integers may be found to satisfy the three-dimensional equation:

$$x^2+y^2+z^2=w^2 \qquad (7)$$

Certain solutions of this equation may be found readily from Table 1. Such special solutions occur whenever one "z" value in the table is equal to another "x" value. For instance, since $3^2+4^2=5^2$ and $5^2+12^2=13^2$, then the former expression for 5^2 may be substituted in the latter equation, giving $3^2+4^2+12^2=13^2$. A few integral solutions of equation (7) are presented in Table 2. All values of x and y listed in this table are found also in Table 1 except that those preceded by an asterisk are multiples of such values. Such solutions can be found indefinitely since for *every* set of x and y

TABLE 2.—SOME EXAMPLES OF INTEGERS SATISFYING THE RELATIONSHIP $x^2+y^2+z^2=w^2$

x	y	z	w	x	y	z	w	x	y	z	w
3	4	12	13	475	132	276	565	429	460	540	829
5	12	84	85	155	468	276	565	455	528	696	985
13	84	132	157	345	152	336	505	*27	36	28	53
77	36	132	157	135	352	336	505	*45	108	44	125
15	8	144	145	171	140	60	229	*45	24	140	149
17	144	408	433	21	220	60	229	*135	72	104	185
143	24	408	433	297	304	168	457	* 63	216	272	353
63	16	72	97	87	416	168	457	*189	180	380	461
33	56	72	97	*375	200	168	457	* 9	12	112	113
323	36	228	397	7	24	312	313	* 9	12	8	17
253	204	228	397	*15	20	312	313	*21	28	12	37
*125	300	228	397	319	360	600	769	*33	44	48	73

values from Table 1, or odd-number multiples of these values, at least one z value can be found to satisfy equation (7), since the successive values of x in the $a=1$ column include all odd integers.

Integral solutions may be found if equation (7) is extended to n-dimensions. For instance, from Tables 1 and 2 we may observe that $3^2+4^2+12^2+84^2=85^2$ or that $3^2+4^2+12^2+84^2+132^2=157^2$, and so on. Any such summation of squares may always be extended to as many more dimensions as desired, at least from values in the $a=1$ column.

It is well known that many other combinations of summations of squares are possible, examples obtainable from Table 1 being $323^2+36^2=253^2+204^2$ and $143^2+24^2=15^2+8^2+144^2$. These combinations are related to one of the interesting theorems in regard to Pythagorean numbers stated by Martin[12] to the effect that "A number that is the product of n different prime numbers of the form $4m+1$ is the common hypotenuse of 2^{n-1} different prime rational right-angled triangles." Of course, other similar combinations of squares consisting of non-Pythagorean numbers, such as $10^2+11^2=5^2+14^2$, may be obtained readily in other ways, but we will not comment further upon such combinations.

All number sets listed in Table 2 are special solutions of equation (7) because the square root of (x^2+y^2), as well as the square root of $(x^2+y^2+z^2)$, is an integer. Other solutions in which (x^2+y^2) is not a perfect square are possible, as in the combination $3^2+2^2+6^2=7^2$, but no attempt was made to tabulate such values.

A special solution of equation (7), for which *all* values of x, y, z, and w occur in the $a=1$ column of Table 1, can be obtained readily from equations (2), the equations attributed to Pythagoras, giving

$$
\begin{aligned}
x &= 2n+1 \\
y &= 2n(n+1) \\
z &= 2n(n+1)(n^2+n+1) \\
w &= 2n(n+1)(n^2+n+1)+1
\end{aligned}
\tag{8}
$$

where n is any integer.

The formulas giving the generally irrational solution of equation (7), corresponding to equations (6) for two dimensions, are rather complicated and of doubtful utility. The way is open for someone to find solutions of equation (7) of a more general nature and to compile a table that is more complete and systematic than Table 2 of such natural rectangular "space quantization numbers."

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Continued on page 40

• APPLICATIONS

Edited by Sheldon S. Myers, Ashville-Harrison High School, Pickaway County, Ohio

Units on four states

Margaret F. Willerding, Assistant Professor of Mathematics at Harris Teachers College, St. Louis, Missouri, submitted some applications with the following comment: "I am enclosing several arithmetic applications for grades 6–9. These units on the states have been successfully used by teachers in the St. Louis Public School System for motivation, drill, and review."

Unit I. Arkansas

1. Arkansas became a territory in 1819 and a state in 1836. The state flag of Arkansas was officially adopted in 1913. Arkansas went without a state flag for ______ years.
2. In size Arkansas stands twenty-sixth among the states, with an area of 53,225 square miles. Of these 810 are water. ______ per cent of Arkansas is water.
3. The growing season ranges from 180 days in the high plateau in the northwest to 240 days in the southeastern part of the state. There is a difference of ______ days of growing season between the southeastern and the northwestern part of the state.
4. Arkansas produces 2,412,000 bushels of peaches and 528,000 bushels of strawberries per year. This is a total of ______ bushels of fruit annually.
5. The Governor's mansion was completed in 1950. The main building is 140 feet by 60 feet. This is an area of ______ square feet.
6. Arkansas mines bituminous coal, bauxite, natural gas, sandstone, limestone, lead, zinc, manganese and iron ore. Arkansas mines ______ minerals.
7. Hot Springs National Park was set aside by Congress in 1832 as a health resort for people of the United States. Hot Springs has been a national health resort for ______ years.
8. More than one million gallons of water flow daily from the 47 springs at the base of Hot Springs Mountain in the park with an average temperature of 143 degrees Fahrenheit. This is an average of about ______ gallons per spring. The 500,000 visitors who annually come to Hot Springs find 300 hotels. If they all came on the same day how many people would be in each hotel? ______
9. The water that comes from the mountain has to be cooled from a very hot temperature. People drink the hot water for their health. Monmouth Spring in Fulton County near the Missouri line has a maximum flow of 150,000 gallons per minute. This is ______ gallons per hour.
10. The two national forests, the Ozark and Ouachita have a total area of 2,293,580 acres. This is an average of ______ acres per forest.

Unit II. Colorado

1. Colorado is called the Centennial State because it was admitted to the Union in 1876, the hundredth anniversary of American independence and ______ years after the discovery of America.
2. Colorado forms a great quadrilateral, and, except for Wyoming, it is the only state that is bounded by four straight lines. Its area is approximately 21 times the area of Connecticut which is 4,965 square miles. The area of Colorado is ______ square miles. It is the seventh state in size.
3. In 1930 Colorado's population was 1,035,791. About 50% of the people live in cities or towns over 2,500, and this urban population is increasing more rapidly than the rural population. The rural areas have about ______ people.
4. Colorado's public school system compares favorably with the best state public school systems in the country. The percentage of illiteracy is low, only 2.8 persons out of every 100 being unable to read and write. Using the 1930 population given above, ______ people cannot read or write.

5. Colorado, in the heart of the Rocky Mountains, has been called the Mountain State. Fifty-one name peaks rise above 14,000 feet; over a thousand peaks reach 72% of this height or ______ feet.
6. Colorado has seven national monuments and the water area of Colorado is about 300 square miles. Recently large petrified trees and stumps have been unearthed in the New Petrified Forest in the heart of the Rocky Mountains. One of the petrified stumps weighs 70 tons. It is 17½ feet in diameter and has a cross section of ______ square feet.
7. The climate of Colorado is usually delightful; the air is dry and the sunshine abundant. In summertime the days are frequently hot, but the nights are cool and bracing. The mean temperature in January is 28.2 degrees; in July it is three times that much, or ______ degrees.
8. Irrigation has developed amazingly in recent years, and everywhere throughout the state are to be seen the great fields with their growing crops. Colorado is second among states in irrigation. Over 33,000,000 acres are now in farms, although only ¼ of this area or ______ acres are actually cultivated.
9. Fourteen national forests lie wholly in Colorado. The total area within these forests and within the borders of the state is 14,748,143 acres. Within recent years, 1,457,789 acres of forest land have passed into private ownership, leaving a net national forest area for the state of ______ acres.
10. The development of its mineral resources is one of Colorado's chief industries and was for a time almost its only industry. Now 14 other states usually rank ahead of it in this industry. Its gold production, which at the beginning of the century was twice that of any other state in the Union, has been exceeded by that of California, and averages only about $7,000,000 a year against $30,000,000 at the turn of the century. What is the per cent of decrease in the last 53 years? ______
11. For many years Colorado has surpassed any other state west of the Mississippi in coal production. The annual production for the 20 years preceding 1926 averaged over 10,000,000 tons, but it has since decreased to about 60% of that average or ______ tons.
12. In Colorado 30,000 trucks and 8 times as many passenger cars, or ______ cars, are registered annually.
13. The constitution under which Colorado is governed dates from 1876, the year of its admission to statehood. Seventeen years later, in ______, one of the most important amendments was passed, permitting women to vote on equal terms with men.
14. Here is a list of people per square mile by decades: 1860—.3 persons; 1870—.4 persons; 1880—1.9 persons; 1890—4.0 persons; 1900—5.2 persons; 1910—7.7 persons; 1920—9.1 persons; 1930—10.0 persons. What is the per cent of increase from 1860 to 1930? ______
15. The Colorado River drains an area of about 225,000 square miles. It is a river of great variations. In some rocky places it is scarcely more than 75 feet wide, but in the open stretches and in the valley region it broadens out to 17.3 times this distance or ______ feet.

Unit III. Massachusetts

When you first visit Massachusetts you are probably most interested in it as a picture book of historical scenes. You may begin either with Cape Cod, the first part of the "Old Bay State" that the Pilgrim Fathers saw and touched in 1620 or with Plymouth, where they made their landing and founded their colony.

1. Plymouth was founded in 1620 and just eight years later, in ______, Salem was founded by John Endicott, and two years later, in ______, Boston was settled.
2. More than 90% of the population (1925 census) of 4,144,205 live in cities or large towns, which is a total of ______ people.
3. In 1633, just ______ years after Plymouth was founded, the colonists began exporting fish.
4. The islands of Martha's Vineyard and Nantucket, the first about 23 miles long and the other 15 miles, making a total of ______ miles, are outlying parts of Massachusetts.
5. The first free public school was founded in Massachusetts in 1635, just ______ years ago.
6. The Mayflower, which brought the Pilgrims across the stormy Atlantic to Plymouth, weighed only 180 tons or ______ pounds.
7. It took the small sailing vessel 63 days or ______ hours to cross the Atlantic Ocean.
8. In 1775, over ______ years ago, the famous ride of Paul Revere was made.
9. Massachusetts runs ⅓ of all the looms and spindles in the United States, and makes 8 miles of cotton cloth a minute, which is a total of ______ yards per minute and ______ miles an hour.
10. Harvard College had its beginning in 1636. The Harvard Medical School occupies a site of 26 acres in Brookline. This is ______ square miles.
11. In 1925 Boston had a population of 780,000 and Worcester 190,000, which is a total of ______ persons for just these two cities.

12. Massachusetts was the home of ______ great inventors, those who invented the cotton gin, the carpet-weaving machine, and the sewing machine.
13. The first printing in Massachusetts was done at Cambridge in ______, just 19 years after the Pilgrims landed.
14. Between 1630 and 1640 a total of 20,000 persons came from England to Plymouth. This is an average of ______ per year.
15. In 1814 in Waltham, ______ years ago, the first power loom was set up in America.

UNIT IV. WYOMING

1. Wyoming has a population of 250,742; 92.2% of its people are native born. This is about ______ persons. Cheyenne is the capital with a population of 22,474. ______% of the state's population lives in the capital. Wyoming has an area of 97,914 square miles. There are ______ persons per square mile. Of its area there are 2,000,000 acres that are irrigated. This is ______ square miles. This makes ______% of the state that has to be irrigated.
2. Wyoming has 26 main highways; half of them join with federal cross-country routes or ______ routes.
3. The first state library was established at Cheyenne in 1870. This was ______ years ago. Now there are 23 library systems which serve 99% of the people. There is a library for every ______ persons.
4. The great plains region is generally level and varies from half a mile to a mile in elevation. This would be from ______ feet to ______ feet.
5. Oil was first discovered in the state in 1832. The first oil well was drilled in 1889. This was ______ years later. Large-scale production wasn't started until after 1910. This was ______ years after oil was discovered and ______ years after the first well was drilled.
6. In 1832 Captain Bonneville and his soldiers blazed the first wagon route across the Rockies. This was used during the Gold Rush in 1850, ______ years later.
7. Between 1840 and 1870 more than 300,000 travelers passed through Wyoming on the overland trails. This was about ______ people per year.
8. Wyoming has 23 counties. It averages ______ square miles per county.
9. The average temperature is 62.7 degrees Fahrenheit in summer and 21.3 degrees in the winter. This is a difference of ______ degrees.
10. Twenty-eight years ago Nellie Ross was elected the first woman governor. She was elected in ______.
11. Glory Hole, of Sunrise Mines, is an open-pit iron mine. About 700,000 tons of ore are shipped from it each year. In ten years the Hole puts out ______ tons.
12. John Colter, a trapper, discovered Yellowstone Park region in 1807. In 1872 the U. S. Government established Yellowstone National Park. It was the first national park in the country. It has been a national park ______ years.
13. The park covers 3,453 square miles in the northwest corner of Wyoming and overlaps Montana and Idaho. This is ______% of the state's area.
14. There are over 200 active geysers in the park. The smallest shoots water 4 feet in the air and the largest 200 feet. This is a difference of ______ feet. Giantess Geyser has been known to spout water for 36 hours or ______ minutes.
15. Old Faithful, best known because it is so regular in its activity, has erupted every 65 minutes since 1870. It shoots silvery cascades 120 to 170 feet in the air and each display lasts 4 minutes. In this time it sends up 10,000 gallons of water at a temperature of 200 degrees Fahrenheit. This is ______ gallons of water per minute. This water is ______ degrees cooler than boiling water.
16. Yellowstone Lake is 15 miles wide, 20 miles long, 300 feet deep, and lies 7,731 feet above sea level. This lake covers ______ square miles.
17. Yellowstone River is 500 miles long and drains an area of 67,500 square miles. The river drains ______ acres.
18. During one vacation season 800,000 persons toured the park in their own cars. One-third of these camped along the way. ______ persons camped during that vacation season.
19. Wyoming was made a state in 1869. This was ______ years ago.

Announcement

Those who wish to exhibit materials made in the classroom at the Thirty-second annual convention at Cincinnati, April 21–24, should contact Mr. Howard Luedeke, Assistant Principal, Walnut Hills High School, Cincinnati, Ohio.

• MATHEMATICAL MISCELLANEA

Edited by Paul C. Clifford, State Teachers College, Montclair, New Jersey, and Adrian Struyk, Clifton High School, Clifton, New Jersey.

"Hail, Hail, the Gang's All Here." The gang in this case being the editors. For like Mother Hubbard, when we got there the cupboard was bare. Just warning you, in case you don't like this department this issue. It's your own fault.

A simple matter of interest

by Paul C. Clifford

In a discussion of interest and savings in a junior high-school class, the question arose, "What procedure is actually followed by local banks in computing interest on savings accounts?" A little investigation disclosed a variety of procedures. Most of the banks credited interest twice a year.

To illustrate procedures, we will assume June 30 to be the end of a period. Figure 1 shows graphically the balance in Mr. C's savings account during this six-month period. We will assume withdrawals and deposits of \$100 units, a uniform interest rate of 2% per year, and each month counted as thirty days.

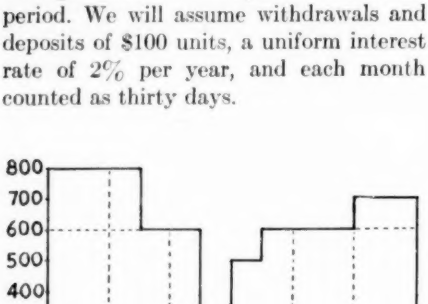


Figure 1

Bank A credited interest on the minimum balance during the six months. Thus the interest would be \$3.00.

Bank B divided the half year into two quarters and used the minimum in each quarter. Thus the interest would be \$1.50 for the first quarter plus \$2.50 for the second quarter, the entire \$4.00 being credited at the end of the half-year period.

Bank C gave interest on the minimum, plus any interest on deposits after the minimum. Complete months only were counted. Thus this bank would credit \$3.00 interest on the minimum, \$1.00 for the deposit made April 1, \$0.33 for the deposit made April 15, and \$0.16 for the deposit made June 1. Total interest would be \$4.49.

As one student summarized it, "Going upstairs you get credit, but not coming down." Also note that the ordinary rules for rounding off numbers do not hold.

Bank D computed interest from the day of deposit or withdrawal. Total interest would be \$6.66. In this bank, interest was actually computed at the time of each deposit or withdrawal, and noted. On June 30 interest was computed on the deposit of June 1. This was totaled with the other partial interests and credited.

The actual procedures varied even more than this summary indicates. Interest rates varied from 1% to $2\frac{1}{2}$%. Some banks counted actual time, others used approximate time. In several banks the only person who would take the responsibility of explaining their system was a vice-president, which would indicate that explanations are requested very seldom.

Theme paper, a ruler, and the hyperbola

by Adrian Struyk

By a method similar to that used for the parabola (see the December, 1953, issue of THE MATHEMATICS TEACHER), points on a hyperbola are easily constructed graphically on composition paper. The two operations involved are drawing straight lines and counting spaces. Points of the curve are obtained as intersections of the construction lines and the rulings on the work sheet.

The procedure to be described arises from the following considerations. In Figure 2 let $A(x_1, y_1)$ be a fixed point in the plane of the axes of reference OX and OY. Let the line through A parallel to OX meet OY at N. On OY let segment TS vary in position, but wherever situated let $TS = NO$, having regard to direction as well as magnitude. Let the line through S parallel to OX meet line TA at a point $P(x, y)$. Then the triangles TSP and TNA are similar. Hence

$$SP : NA = ST : NT.$$

But

$$SP = x, \quad NA = x_1, \quad ST = ON = y_1$$

and

$$NT = OS = y.$$

Thus

$$x : x_1 = y_1 : y,$$

so that

$$xy = x_1y_1 = \text{a constant}.$$

Now the equation of a hyperbola, referred to its asymptotes as axes, is $xy = k$, k a constant. Therefore the locus of P is the hyperbola which passes through A and has asymptotes OX and OY.

The geometrical conditions imposed upon the lines and points in Figure 2 are just those which composition paper, with its equally spaced parallel rulings, enables us to achieve so readily. Applying the considerations of the preceding paragraph we proceed as indicated in Figure 3. One of the rulings on the work sheet is taken as OX. Point A is marked on a ruling other than OX. OY is a line which cuts the rulings and does not pass through A. The admissible positions of T are the intersections of OY and the rulings. For each T there is a corresponding P, the intersection of line AT and a particular ruling. The correct ruling for a given T bears the same relation to T in regard to spacing and di-

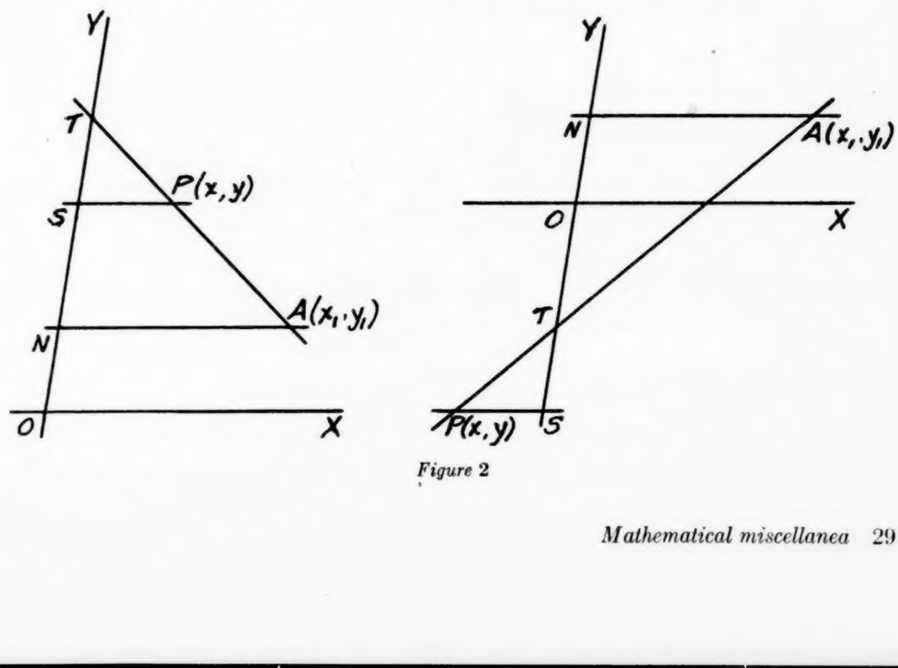


Figure 2

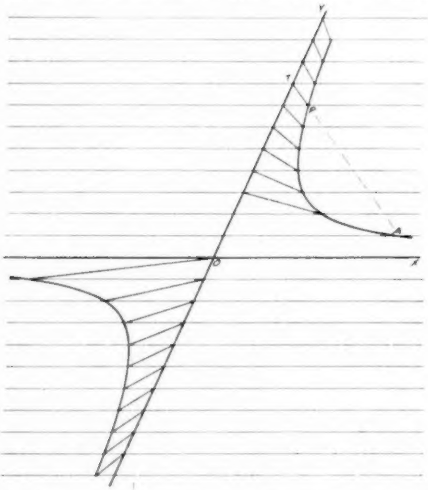


Figure 3

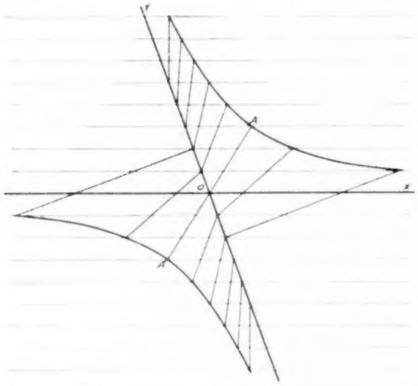


Figure 4

rection as OX bears to A. The process of determining the points of the curve may be described as "central projection." From the "center" A each T is "projected" upon the correct ruling.

Figure 4 shows a slight variation which achieves a more effective use of the rulings near one end of the work sheet. After selecting A reflect it in O, obtaining A' such that O is the midpoint of AA'. Then project from A' to obtain points on the hyperbolic arc through A, and project from A to determine points on the hyperbolic arc through A'.

Cross-figure puzzle

Contributed by Jessie R. Smith, St. Cloud, Minnesota

ACROSS

1. Eleventy.
4. Retirement age.
6. Mrs. Lotta Kids had two pair of twins twice. How many children in all?
8. Daffy definition of a vacation: Two months, after which you are ______ tired ______ work and ______ broke not ______.
10. When Noah sailed the ocean blue
    He had his troubles same as you;
    For ______ days he sailed his ark
    Looking for a place to park!
12. The product of the number of teeth in a bald man's comb and the number of teeth in his head.
13. He who would thrive must rise at ______;
    He who has thriven may lie till ______.
14. A very naughty number!
16. How Miss Birdbrain wrote "thirty-one."
17. One owes nothing for one ate nothing.
21. A man went hunting and shot a jay-bird. He and his wife ate it for supper. What was their telephone number? Ans. ______J.
22. A centipede with corns would pay how much for pads at $2\frac{1}{2}$ cents each? Ouch!
23. "The bride was escorted to the altar by tight bridesmaids and sour flower girls." What slipped?
25. My top and my bottom are round as a ball
    But without my top, I am nothing at all.
26. Riddle: The only number more than ten that rhymes with a number less than ten.
28. Four score and seven years ago in 1951.
30. I love you a bushel and a peck
    How many quarts is that, by heck?
32. Definition of a duel: Pistols for ______; breakfast for ______.
34. History: ______ thousand Swedes went through the weeds, chased by ______ Norwegians.

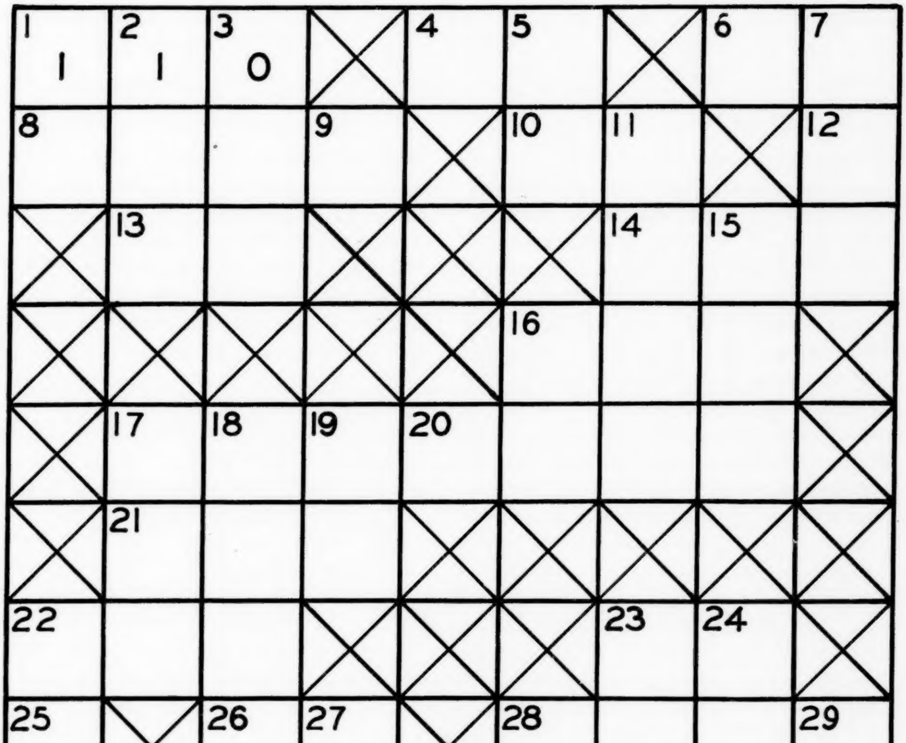
DOWN

1. Munch hour.
2. A sweet story: Take 125 jelly beans from 250 and how many have you?
3. Rhymes with "Go to heaven"!
4. Riddle: What number increases by one-half when inverted?
5. If a dog weighs 54 pounds when standing on

three feet, what will it weigh when standing on all four?

6. For men only: A bachelor is a man who doesn't make the same mistake ______ times!
7. The army that rode into the Valley of Death.
9. He named his ship "Canasta" because it had ______ decks.
11. "Psychologists say that pupils remember about 10% of what they hear, about 35% of what they see and about 90% of what they do." If false, write 0118; if true, write 0008, and start a Mathematics Work Room!
15. Half of 818. Hint: Take the top half!
16. On April Fool's Day I am always tired after a March of ______ days.
17. You will have wheels in your head if you can compute the total number of traction wheels: a wagon loaded with 4 wheelbarrows, three of the wheelbarrows each being loaded with 9 bicycles and 7 tricycles.
18. The next term in the series: 3531, 2621, 1711,
19. None won.
20. The middle half of FIVE.
22. CCLXXXIV
23. The report read, "881 dead." Drop dead.
24. 920 and 2. D'vide, d'faster d'better.
27. Said the teacher as she shot her husband, "I have learned that ______ can live more cheaply than ______.
28. If it takes 1 minute to boil 1 egg, how long will it take to boil 3 eggs?
29. What is the difference between a Lincoln penny and the new fifty-cent piece?
31. The area of a triangle whose sides are 7, 3, and 2. If you don't know, guess.
33. A number that rhymes with none.

(See answer on page 52.)



Edited by Emil J. Berger, Monroe High School, St. Paul, Minnesota. Anyone who has a learning aid which he would like to share with fellow teachers is invited to send this department a description and drawing for publication. If that seems too time-consuming, simply pack up the device and mail it. We will be glad to originate the necessary drawings and write an appropriate description. All devices submitted will be returned as soon as possible. Send all communications to Emil J. Berger, Monroe High School, St. Paul, Minnesota.

Using a carpenter's folding rule to teach the meaning of perimeter

Contributed by Neil L. Gibbins, Olmstead Falls High School, Olmstead Falls, Ohio

A device that can be used as an aid for giving meaning to the term *perimeter* and the formula for the perimeter of a rectangle is the carpenter's folding rule.

Slower pupils many times confuse the formulas for perimeter and area of a rectangle, and when asked to find the perimeter insist on multiplying the length by the width instead of finding the sum of the lengths of the four sides.

By using the simple device shown in Figure 1 pupils are able to see at a glance that one must add in order to find the perimeter, because the sides are automatically summed as one proceeds around the figure. Straightening out the rule also helps clarify the meaning of perimeter as contrasted with the concept of area.

Some teachers may think that the physical properties of the rule may cause pupils to get the idea that a line has width, but this is not the case. Of course this method of instruction should not be allowed to become the sole means of presenting the concept of perimeter as it relates to a rectangle, but rather it should be used as an aid for assisting the slow learner who has difficulty grasping the idea.

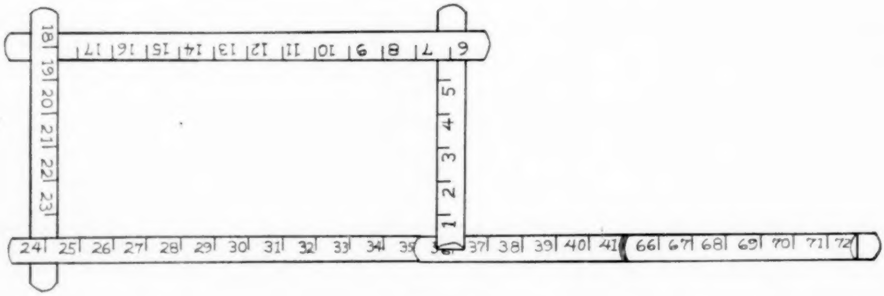


Figure 1

A model for giving meaning to superposition in solid geometry

By Emil J. Berger

We trust that our introducing this solid geometry model will not involve us in any controversy about whether superposition should be used in developing certain congruence relations. However, we truthfully confess that the inclusion of this particular model has long been postponed because it was felt that no sensible discussion of its use could be presented without meeting the question about superposition, and that to join in that issue would carry us beyond the province of this department.

We propose a model for use in proving the theorem which may be stated as follows:

Two prisms are congruent if three faces which include a trihedral angle of one are congruent respectively to the three faces which include a trihedral angle of the other, and are similarly placed.[1]

The proof of this theorem as presented in most standard textbooks on solid geometry depends, of course, on the use of the postulate of superposition. The usual statement from a typical proof is this one: "Place prism P' on prism P so that trihedral angle A' coincides with trihedral angle A."

Now one of the objections to the use of superposition in proving congruence is that an actual figure cannot be moved because it is made up of points and lines (each of which in turn is determined by two points). The argument is that since a point has neither length, breadth, nor thickness, but only the property of position, it cannot be moved; hence geometric figures determined by such points cannot be moved.

Another objection to the use of superposition takes this form. In terms of our theorem we actually assume that it is possible to move a prism when we know only that three of its faces which include a trihedral angle will not change in shape or size during such an operation. This means that three faces which include a trihedral angle must determine the prism—in other words, insure its size and rigidity. Thus when one bases his proof on the superposition postulate he really assumes congruence at the outset and the remainder of the proof is empty.[2]

Logically speaking these objections are irrefutable, and the usual way out is to postulate those congruence theorems which depend for their proofs on the principle of superposition. However, one cannot make the concept of congruence plausible to students by mere postulation of the fact that under certain conditions two figures are said to be congruent. Furthermore, such procedure, if carried out consistently, would reduce an important part of the subject to pure assumption where much more understanding may be had if some type of "demonstration" is provided. In solid geometry particularly much of the meaning inherent in different concepts depends on the relations a student sees in them, and such understandings are best developed by working out proofs for the more important theorems even if such proofs may have to be semiformal.

So while superposition may not be logically sound, it is certainly not pedagogically unsound to develop a proof by accepting it temporarily as a method—at least for students who are capable of studying solid geometry. In fact this may be a good place to introduce them to a

[1] This statement of the theorem is taken from the textbook, *Solid Geometry (Revised Edition)*, by A. M. Welchons and W. R. Krickenberger (New York: Ginn, 1950), p. 86.

[2] For a brief discussion of the criticisms against the use of superposition see *Geometry Professionalized for Teachers*, by H. C. Christofferson (Oxford, Ohio: the author, 1933), pp. 35–38.

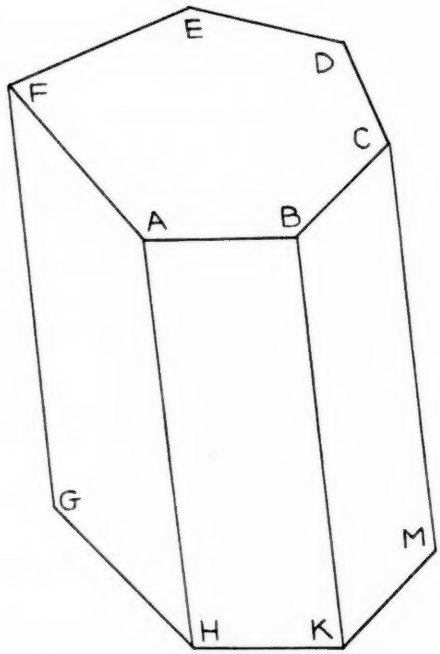
proof which depends on the use of superposition, and then to point out the objections to its use. An opportunity like this is worthwhile in itself, because it gives students a real insight into the logical structure of geometry as a system of reasoning.

Suppose then that superposition is accepted as a method for proving the theorem stated above. What meaning can be attached to the maneuver of moving concrete figures into coincidence? The device described in this article was designed especially to give *some* meaning to this act.

The device we have in mind for developing a proof of the theorem stated above is composed of two separate parts, one of metal and the other of wood. The plan of construction is simple, but effecting the production requires a little more skill than is generally the case with devices that are described in this section. Actually this device can also be made of paper or plywood, but for some strange reason the combination metal and wood model which we suggest gets more attention from students than ones made from simpler materials.

Our model is a heavy affair, and it is absolutely rigid. Students who handle it invariably comment that now they know what is meant by the statement, "A geometric figure may be moved without changing its shape or size."

The wooden part, prism P in Figure 2, is a solid piece of wood made by gluing together two blocks each $5'' \times 5'' \times 8''$ and then sawing the composite block obliquely across the grain of the wood twice and six times along its length in such a way that the resulting solid is an oblique hexagonal prism. Note that the bases in Figure 2 are irregular hexagons; sawing the block to this shape helps add a note of generality to the figure. The lateral edges of prism P are all $6''$, and the longest diagonal of each base is also $6''$. Unfortunately it is not very helpful to include the lengths of the sides of the bases since these depend on the angle of obliquity of the prism.



PRISM P

Figure 2

The easiest way to saw the block to the shape described is to use a power saw, but the task is by no means impossible if one must rely on a hand saw. However, if one has only a hand saw, the prism should be made of soft wood. To complete prism P paint its entire surface in some neutral color, and then paint the faces FH and AK red and yellow respectively and the base $ABCDEF$ some other color, say green. If letters are stenciled on the surface as indicated they should be done in black.

To make the second part of the model, cut out two pieces of sheet metal to correspond with the shapes illustrated in Figures 3 and 4. $A'B'C'D'E'F'$ must be congruent to $ABCDEF$. The flanges along edges $A'B'$ and $F'A'$ should each be about $\frac{3}{8}''$ wide. Cutting the dihedral angle $F'-A'H'-K'$ so that its faces will coin-

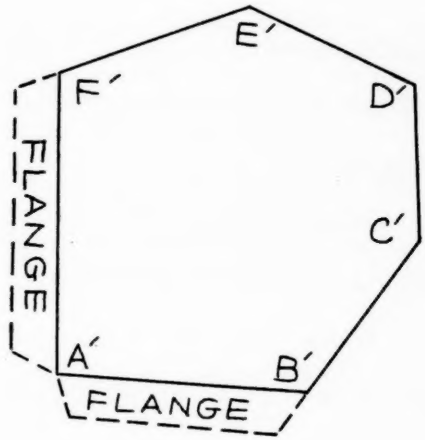


Figure 3

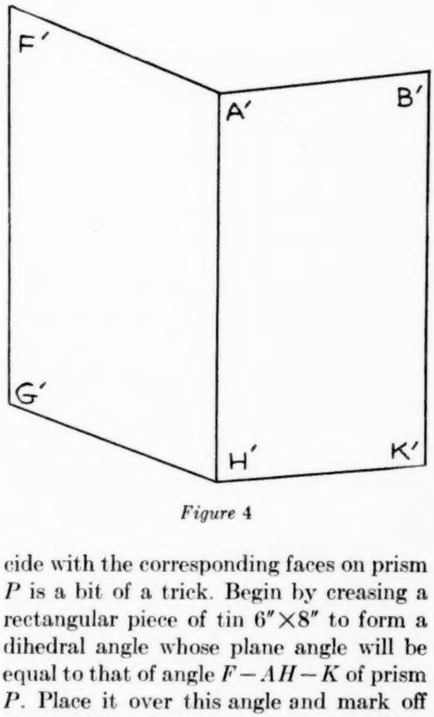


Figure 4

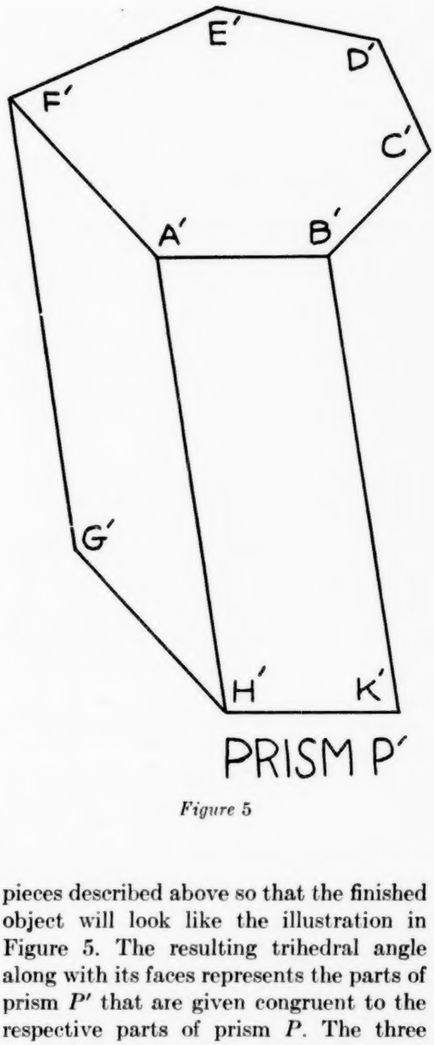


Figure 5

cide with the corresponding faces on prism P is a bit of a trick. Begin by creasing a rectangular piece of tin $6''\times 8''$ to form a dihedral angle whose plane angle will be equal to that of angle $F-AH-K$ of prism P. Place it over this angle and mark off faces $F'H'$ and $A'K'$ with a pencil. When the resulting pattern is cut out with a tinner's shears it should look like the illustration in Figure 4. To complete this part of the model (which we shall refer to as prism P') solder together the two metal pieces described above so that the finished object will look like the illustration in Figure 5. The resulting trihedral angle along with its faces represents the parts of prism P' that are given congruent to the respective parts of prism P. The three faces should be painted and lettered with primed letters to correspond with the colors and letters of prism P. From this point on the use that may be made of the model should be obvious.

Whether the use we suggest for this model is logically sound, or whether it should be used at all are questions which we leave to the judgment of the reader, but experience seems to be on our side in that as an aid to visualization this model is a real help.

Edited by Phillip S. Jones, University of Michigan, Ann Arbor, Michigan

The quadrature of the parabola: an ancient theorem in modern form

Contributed by Carl Boyer, Brooklyn College, Brooklyn, New York

The quadrature of the parabola by Archimedes is one of the best-known of the classics of the history of mathematics. His familiar squaring of the parabola depends on the following property: Let P_1, P_2, P_3, P_4, P_5 be points on the parabola $y^2=2px$ such that P_2M_2, P_3M_3, P_4M_4 are parallel to the axis, where M_2 is the midpoint of P_1P_3, M_4 is the midpoint of P_3P_5, and M_3 is the midpoint of P_1P_5. Then $\triangle P_1P_2P_3 + \triangle P_3P_4P_5=\frac{1}{4}\triangle P_1P_3P_5$. Numerous demonstrations of this property are available; but the following proof seems to be more expeditious and should serve as an appropriate classroom exercise in analytic geometry.

Making use of determinants, we have

$$\Delta P_1P_2P_3=1/2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$=\frac{1}{4p}\begin{vmatrix} y_1^2 & y_1 & 1 \\ y_2^2 & y_2 & 1 \\ y_3^2 & y_3 & 1 \end{vmatrix}$$

$$=\frac{1}{64p}\begin{vmatrix} y_1^2 & y_1 & 1 \\ (3y_1+y_5)^2 & 4(3y_1+y_5) & 16 \\ y_3^2 & y_3 & 1 \end{vmatrix},$$

using the equation of the parabola $y^2=2px$ and the point-of-division formula

$$y_2=\frac{3y_1+y_5}{4}.$$

Remembering the midpoint formula

$$y_3=\frac{y_1+y_5}{2},$$

one subtracts from the elements of the second row twelve times the corresponding elements of the last row (in order to eliminate the cross-product term in the first column) and six times the corresponding elements of the first row, obtaining

$$\triangle P_1P_2P_3=\frac{1}{64p}\begin{vmatrix} y_1^2 & y_1 & 1 \\ -2y_5^2 & -2y_5 & -2 \\ y_3^2 & y_3 & 1 \end{vmatrix}$$

$$=\frac{1}{32p}\begin{vmatrix} y_1^2 & y_1 & 1 \\ y_3^2 & y_3 & 1 \\ y_5^2 & y_5 & 1 \end{vmatrix}$$

$$=\frac{1}{16}\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_5 & y_5 & 1 \end{vmatrix}$$

$$=\frac{1}{8}\triangle P_1P_3P_5.$$

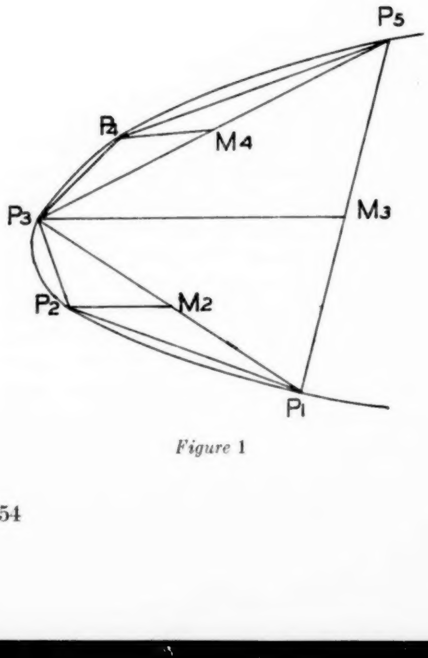


Figure 1

Noting that the area of $\triangle P_3P_4P_5$ is obtained by interchanging the subscripts 1 and 5 where they occur above, one has $\triangle P_3P_4P_5 = \triangle P_1P_2P_3$; and hence $\triangle P_1P_2P_3 + \triangle P_3P_4P_5 = \frac{1}{4} \triangle P_1P_3P_5$.

Continuing the subdivision process in the usual Archimedean manner, and applying the above property recursively, one obtains the classic result—that the parabolic segment $P_1P_3P_5$ is given by $\triangle P_1P_3P_5\,(1+\frac{1}{4}+1/16+\cdots\frac{1}{4}n+\cdots) = 4/3\,\triangle P_1P_3P_5$.

Editor's Note: On the chance that some of our readers are not as familiar with Archimedes' (230 B.C.) work as Professor Boyer assumes, we point out that this procedure uses the "method of exhaustion" credited to Eudoxus (350 B.C.) and so named by Gregory St. Vincent in 1647. Archimedes used only implicitly the limit of the sum of an infinite geometric progression. Both area and series problems are related to the calculus. The discovery in 1906 of Archimedes' lost work on *The Method* revealed how amazingly close to modern calculus he had been in the procedures he used to discover these results which he then recast into a classical and acceptable "method of exhaustion" form before "publishing" them to his contemporaries. Further details on these procedures and techniques would be found in Sir T. L. Heath's *Works of Archimedes,* recently reprinted by Dover Publications, or Professor Boyer's own *Concepts of the Calculus* (New York: Hafner Publishing Co.).—PHILLIP S. JONES.

Geometric progressions in America and Egypt

Contributed by Norman Anning, 909 Mt. View Terrace, Alhambra, California

Professor Anning sends us the page from a student's copy book which is shown in Figures 2 and 3. The book was titled:

Timothy Street's

Book

Middle Road, Trafalgar, Feb. 17th, 1841

Professor Anning adds the following comments:

Timothy Street was a distant relative.

I believe "Trafalgar" was about 30 miles northwest of Toronto, C.W. (Canada West).

Please overlook the pounds, shillings, and pence (Canadians still computed in £.s.d., but then, as now, were happy to accept Yankee $$$) and observe the clumsy way in which the student finds the "some" of a geometrical progression

$$1+4+4^2+\cdots+4^{11}.$$

He uses a formula, old as Ahmes, I believe, to sum eleven terms and then adds on the twelfth term.

Over the page is the old problem about the "nales" in the shoes of a "hors"; "there were four shoes"! The student blunders in computing the value of 3^{11} but still arrives at the *correct* price of the horse! No comment.

For a check:

$$1+3+3^2+\cdots+3^{31}=\frac{3^{32}-1}{3-1}$$

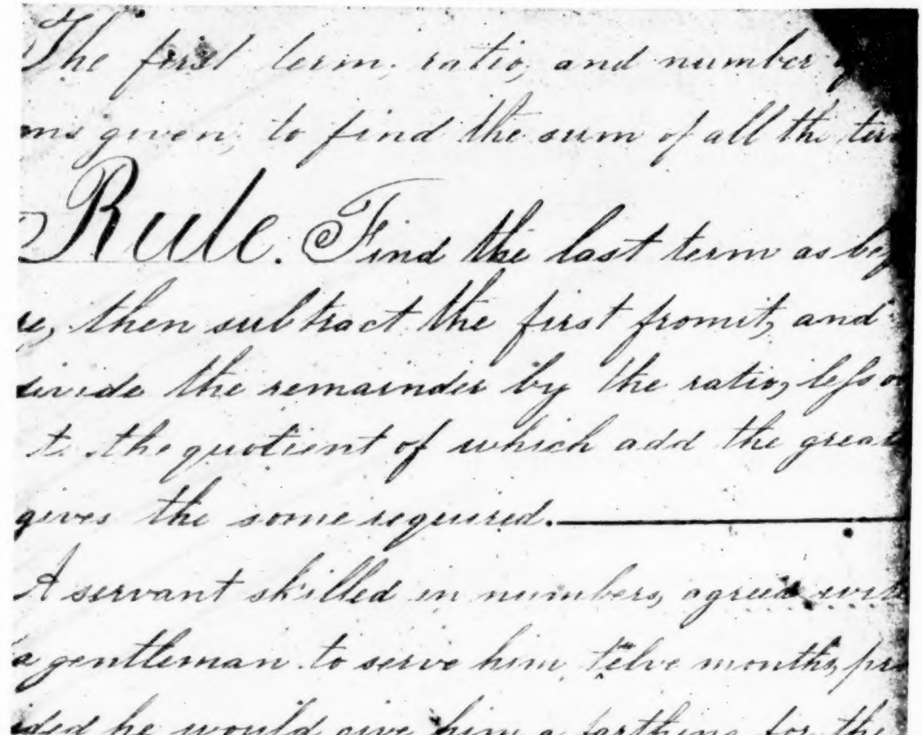
$$\begin{array}{r|l} 2 & \underline{1853020188851840} \\ & \;926510094425920 \end{array}$$

Note that $3^{32}=81^8$ and can be lifted from page 255 of the 1941 edition of *Barlow's Tables.*

Editor's Note: Professor Anning no doubt refers to Problem 79 of the Ahmes (or Rhind) Papyrus which dates back to about 1550 B.C. This problem arrives at the same result by two different procedures. One column translated reads

houses	7
cats	49
mice	343
spelt	2401
hekat	16807
total	19607

There have been several interpretations of this, but a favorite one is that each of seven houses had seven cats, each of which caught seven mice which would have eaten grain. The story behind the last two quantities is less clear (a hekat is a measure of grain). A similar problem appears in

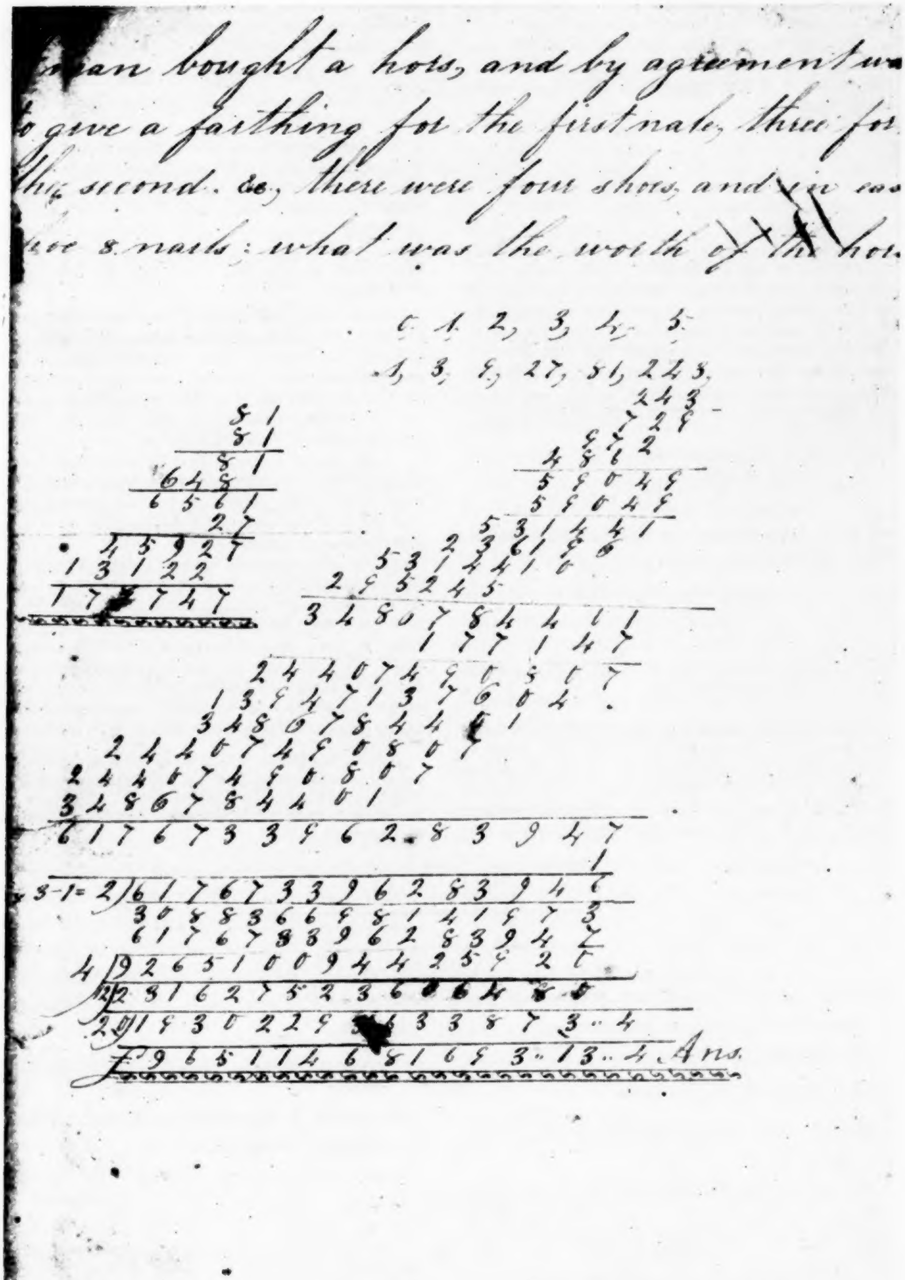

The first term, ratio, and number
ms given, to find the sum of all the te
Rule. Find the last term as be
e, then subtract the first from it, and
ivide the remainder by the ratio, less o
t. the quotient of which add the grea
gives the sume required.
A servant skilled in numbers, agreed wit
a gentleman to serve him twelve months, pr
ided he would give him a farthing for the
first monthes service, a penny for the seco
and 4d. for the third, &c., what did his wages amo

0, 1, 2, 3, 4,
1, 4, 16, 64, 256

256
256
1536
1280
512
65536
64
262144
393216
4-1=3)4194304
1398101

4194304
1398101
4)5592405
12)1398101¼
20)116508..5¼
£5825..8..5¼ Ans

Figure 2



man bought a hors, and by agreement wa
to give a farthing for the first nale, three for
the second. &c, there were four shoes, and in ea
shoe 8 nails: what was the worth of the hor

0. 1. 2, 3, 4, 5.
1, 3, 9, 27, 81, 243.

81
81
81
648
6561
27
45927
13122
177147

3-1=2)617673396283946
308836698141973
617673396283947

£9 6 5 1 1 1 4 6 8 1 6 9 3..13..4 Ans.

Figure 3

Leonard of Pisa's *Liber Abaci* (1202). To moderns both problems associate with the nursery rhyme-puzzle, "As I was going to St. Ives, ..."

But, returning to the Egyptian problem, the second column shows the product of 2801 by 7 using the doubling system of Egyptian arithmetic thus

1	2801
2	5602
4	11204
Total	19607

2801 is one more than the sum of the first four terms of the progression. Hence some people think that the Egyptians knew the rule for the sum of geometric progressions *in which the first term and the common ratio, r, are the same;* namely, the sum of n terms is the sum of $n-1$ terms plus one, all multiplied by r. In symbols this would be $S_n=(S_{n-1}+1)r$

Timothy Street's rule in symbols would be

$$S_n=\frac{u_n-u_1}{r-1}+u_n$$

where u_n and u_1 are the last and first terms. Since $u_n=u_1^{n-1}$,

$$\frac{u_n-u_1}{r-1}=S_{n-1}$$

and $S_n=S_{n-1}+u_n$. This is a general formula without Ahmes' restriction.

Returning to America, one of the most curious books of the last century was *The Poetical Geography with the Rules of Arithmetic in Verse* copyrighted in 1849 by John Street (no relation to Timothy so far as I know). The 1861 edition owned by the editor has pages divided vertically with verse on the left side of each page and a prose translation or restatement on the right side. The poetical portion of the last item on the last page (p. 96) reads:

Geometrical Progression

The first term, ratio, and number of terms being given, to find the last term.

A few leading powers of the ratio write down,
With each index placed o'er, beginning at *one,*
The *indices* whose *sum* as the rule thus informs,
Shall approach within *one* of the number of terms,
Stand over the factors, whose product must be
Multiplied by the *first* term, and the last term we see.

No rule for the sum of the geometric progression was given in this book, but Nicholas Pike in his A NEW AND COMPLETE SYSTEM OF ARITHMETIC COMPOSED FOR THE USE OF CITIZENS OF THE UNITED STATES, "the recognized American arithmetic from 1788 well into the nineteenth century"[1] states:

Rule: Raise the ratio to a power, whose index be equal to the number of terms, from which subtract 1; divide the remainder by the ratio less one, and the quotient multiplied by the first term will give the sum of the series.

This, of course, is essentially Timothy Street's rule. Not only is it interesting to note something of the content of the "arithmetic" of an earlier day, but the verification of these two rules is good problem material which can be made to grow out of the interest-stimulating stories of Egyptian mathematics and the history of progressions. Archimedes' summations are also a part of the latter.— PHILLIP S. JONES.

[1] L. C. Karpinski, *The History of Arithmetic* (Chicago: Rand McNally, 1925), p. 83. The first edition of Pike was published at Newburyport. The rule is copied from page 239 of the second edition published at Worcester in 1797.

Periods of service of the officers of the National Council of Teachers of Mathematics

Continued from page 24

Henry Van Engen, Cedar Falls, Iowa, 1951–52

Allene Archer, Richmond, Virginia, 1952–

Ida May Bernhard, Austin, Texas, 1952–

Harold P. Fawcett, Columbus, Ohio, 1952–

Houston T. Karnes, Baton Rouge, Louisiana, 1953–

Howard F. Fehr, New York, New York, 1953–

Phillip S. Jones, Ann Arbor, Michigan, 1953–

Elizabeth J. Roudebush, Seattle, Washington, 1953–

Editorial Note: This list of names published in compliance with a ruling of the Editorial Board, namely: Lists of names shall not be published in THE MATHEMATICS TEACHER without the approval of the Editorial Board.

• POINTS AND VIEWPOINTS

Introducing a new department that will present the views of the President and official members of the NCTM

As you have already noted, this issue of The Mathematics Teacher introduces a new design, and there are some other features which are new to readers. Why the changes? The Editor feels that readers are entitled to know the reasons for the changes in their magazine.

Why "Points and viewpoints"?

These columns will replace the "President's Page" so familiar to the readers. Changing the title of these columns does not materially change the purpose of the columns. They will always remain open to the President of the National Council of Teachers of Mathematics at any time and all the time if the President so desires. In such cases the President will have the opportunity to speak for the organization and to present points of view which he feels are vital to the Council and to the welfare of mathematics education. In other instances, as in this issue, these columns will be open to other members of the official family of the Council as well as to leaders in the field of mathematics education to present the "topics of the day" or to call the readers' attention to matters of importance.

The new design

Why the change in design? E. H. C. Hildebrandt, when he resigned as editor, recommended that the covers of the magazine be redesigned because they were too crowded. Hildy thought they needed to be "opened up." The present editor thought this was a good idea, but, as so frequently happens with good ideas, the matter of redesign did not stop with the cover. Why not attempt to freshen up the whole magazine? Not that it was necessarily stuffy—Hildy did too good a job for that—but it might be possible to improve even a good magazine.

With this thought in mind, the problem was taken to those who know something about magazine designs. Then after many hours of conference and a few more hours of thought the result appears as you see it. What is back of the design? An attempt has been made to make the reader see this magazine as made up essentially of four kinds of material: (1) the feature articles, (2) the departments, presenting original contributions, (3) the reviews and evaluations of published material and manufactured material, and (4) the material dealing with the affairs of the National Council of Teachers of Mathematics. Each of these four sections of the magazine will have its own stylized heading, and in some cases sections will be set apart from the others by means of a change in type size.

The cover has been redesigned and opened up as Hildy recommended. The Editor hopes that the index of the feature articles on the front cover will make you want to look further; that seeing the complete index on the first page will give you a good picture of what is in store for the reader, and that the inside front cover will tell you, at a glance, who are the officers of your organization.

Changes in departments

New departments have appeared and will appear from time to time. "What's

New" will tell you quickly about the new publications, new films, new filmstrips, new equipment, and other new products which will be of interest to mathematics teachers. Later, in other issues of the magazine, these items will be reviewed in the "Reviews and Evaluations" section of the magazine. Other new columns will include "Tips for Beginners," edited by Francis Lankford of the University of Virginia. Others may be announced in the future.

Some old friends wish to be relieved. The well-known and highly respected section, "Aids to Teaching," is not in this issue. Henry Syer and Donovan Johnson have done a magnificent piece of work for teachers of mathematics in this department. However, they feel that since much of the material they have presented in the past deals with arithmetic and since this kind of material should now be presented in our companion publication *The Arithmetic Teacher*, there will probably be no need for a monthly "Aids to Teaching." Furthermore, since they are now vitally concerned with other projects they have requested that new editors be found for their department before next October's issue comes off the press.

The Editor regretfully released these two men from their responsibility. The readers of the magazine will miss their contributions. We wish them the best of luck in their new ventures. In order that their work may be carried on for the readers, their department will be incorporated in the not too distant future with the "Reviews and Evaluations" section of the magazine.

The Arithmetic Teacher

Some changes are necessitated because the National Council of Teachers of Mathematics is now represented by two magazines: THE MATHEMATICS TEACHER and *The Arithmetic Teacher*. THE MATHEMATICS TEACHER will devote its attention to the grades seven through fourteen and to secondary teacher education. *The Arithmetic Teacher* will be interested in the kindergarten and the six grades of the elementary school and in elementary teacher education. For this reason articles on arithmetic will disappear from THE MATHEMATICS TEACHER. Also that part of the "Book Section" and "Aids to Teaching" devoted to elementary-school teaching will be deleted from THE MATHEMATICS TEACHER.

Ben Sueltz, who has served with Cecil Read as editor of the "Book Section," will now devote his time and thought to the problems of the editor of *The Arithmetic Teacher*. Ben will do a good job and the readers will want to wish him the best of luck as well as offer him a hand for doing those little things that each of us can do.

The Editor of THE MATHEMATICS TEACHER hopes that you will like these changes in the magazine. If you like them, he will welcome your saying so, but it is even more important for you to let the Editor know about the changes which you do not like and why you don't like them. The Editor and the Editorial Board have no way of judging whether you like the magazine unless you let them know. Silent partners may be ideal in business, but editors don't like silent readers.

Microfilm edition of The Mathematics Teacher available

The issues of THE MATHEMATICS TEACHER beginning with Volume XLVI are now available on microfilm. Libraries and others with storage problems will welcome this new service. The microfilm edition will include all the issues in one volume and will be available at the close of the volume year, which ends with the December issue. Sales will be made only to subscribers and members of the Council. Orders and inquiries should be sent to University Microfilms, 313 N. First Street, Ann Arbor, Michigan.

Edited by William L. Schaaf, Department of Education, Brooklyn College, Brooklyn, New York

Science, Mathematics and Religion

"I cannot believe that God plays dice with the cosmos!"

—*A. Einstein*

"The Great Architect of the Universe now begins to appear as a pure mathematician."

—*J. H. Jeans*

In an age of personal tensions and social uncertainties such as the present, a revival of general interest in religious matters occasions no great surprise. And that is precisely what the Western World is now witnessing. Much has been written over the years on the "conflict" between science and religion, and upon the "resolution" of this conflict; somewhat less has been said about the relation between mathematics and religion. It is to be hoped that bringing together the following source material will prove helpful.

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2. Humanism and Mathematics

"He who knows mathematics and does not make use of his knowledge, to him applies the verse in Isaiah (v. 12), 'They regard not the work of the Lord, neither consider the operation of his hands.' "

—*Talmud*

"The laws of nature are but the mathematical thoughts of God."

—*J. Kepler*

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3. Religion and Mathematics

"There is no prophet which preaches the superpersonal God more plainly than mathematics."

—*Paul Carus*

". . . what is physical is subject to the laws of mathematics, and what is spiritual to the laws of God, and the laws of mathematics are but the expression of the thoughts of God."

—*Thomas Hill*

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4. THE INFINITE IN MATHEMATICS

"The infinite! No other question has ever moved so profoundly the spirit of man."
—*David Hilbert*

Mathematicians have been wrestling with the concept of infinity for more than two thousand years, from the days of Zeno and Democritus to the present-day uncertainties of mathematical philosophy. The trail has been a long one, beset with aggravating contradictions and baffling paradoxes. Nor has the last word as yet been said—perhaps there is no last word. Whether it be the infinite in algebra and analysis, the infinite in geometry, or the hierarchy of transfinite numbers in the theory of aggregates, the subtleties as well as the sweeping implications of the concept stagger the imagination.

We cannot escape the infinite in mathematics. It is fundamental to the very concept of number; it lies at the basis of the concept of space; it is inextricably bound up with the idea of motion; modern mathematical analysis would be trivial without it; the notion of infinite sets has split the field of mathematical philosophy into rival camps and shaken the logical foundations of mathematics. Its impact upon physical science has been scarcely less profound, and the implication of the infinite for religious belief needs no apology.

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• TIPS FOR BEGINNERS

Edited by Francis G. Lankford, Jr., University of Virginia, Charlottesville, Virginia

Checking the answers is not enough

CONSIDER THE case of an eighth-grade girl, with somewhat above average ability, who took a test on a review of fractions which included these three items.

$$\text{(a)}\ \begin{array}{r} 3/4 \\ \underline{5/8} \end{array} \qquad \text{(b)}\ 5/6+2/3=?$$

$$\text{(c)}\ 3/5+7/10+1/2=?$$

Correct answers were obtained to (a) and (c) and the incorrect answer $1\frac{1}{4}$ was given for (b). An interview was arranged to "think aloud" the steps she had taken in these addition exercises. For (a) she arranged her work this way and explained it quite mechanically.

$$\begin{aligned} 3/4 &= 6/8 \\ 5/8 &= \underline{5/8} \\ &\quad 11/8 = 1\ 3/8 \end{aligned}$$

For (b) she arranged her work this way

$$5/6+2/3+5/\underset{2}{\cancel{6}}+\cancel{3}/2=5/4=1\ 1/4$$

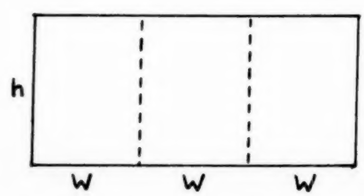
and explained the steps equally mechanically. She even wrote a plus sign for the "invert-and-cancel" step. For (c) she rearranged the fractions for vertical addition.

$$\begin{aligned} 3/5 &= 6/10 \\ 7/10 &= 7/10 \\ 1/2 &= \underline{5/10} \\ &\quad 18/10 = 1\ 8/10 = 1\ 4/5 \end{aligned}$$

When asked why she operated with (b) as she did, she replied, "That's the way you do that kind." Then she was asked why she did not do (c) as she did (b). Her answer was, "You have three fractions and you can't tell which one to invert."

It seemed rather clear to the interviewer that to this girl arithmetic was a matter of manipulating algorisms. Indeed, the arrangement of an exercise on a page was used as a clue in deciding which algorism to use. The interview revealed as little understanding of the two exercises for which correct answers were found as the one for which an incorrect answer was found.

Many other similar experiences with "diagnostic interviews" have suggested that any teacher of mathematics will do well to find the time every so often to sit down with a student individually and have him indicate orally what his ideas are on mathematical concepts, principles, and operations. Some needless reteaching may be done unless the *real* reasons for a student's faulty work is known. Moreover, individual interviews will often yield surprising explanations of incorrect work. A high school senior was asked on a general achievement test to write a formula for finding the length of picture framing needed for a frame used to hold three pictures as in this drawing. Her answer was $4h+6w$. When questioned in an interview



she explained that she had a frame like this at home and that her frame had wooden divisions at the broken lines in the drawing. She, therefore, answered the question according to the picture frame she had rather than the one in the drawing.

Such interviews are as helpful with the conscientious bright pupil who seldom gets a wrong answer as with the pupil of lower ability who makes many mistakes. The bright pupil often gets his correct answers with much too great dependence upon tedious and mechanical manipulations. It is important that such a pupil be encouraged to free himself of such fetters to thinking. He is often as much in need of corrective teaching as is his slower classmate.

Such a bright seventh-grade boy (I.Q. 122) showed this tendency, for example, when he was questioned about his answer of 4 as the product of this exercise.

$$\begin{array}{r} 3/4 \\ \underline{\times 1/3} \end{array}$$

He explained his thinking this way: "Cancel 3 into 3 and this leaves 4." When questioned further he replied, "That's not the way we do them." He was then asked to find the product in his accustomed manner. Quickly he rearranged the exercise like this: $\frac{3}{4}\times\frac{1}{3}$ from which he obtained a product 3/12 and which he reduced to $\frac{1}{4}$. Surely a pupil with I.Q. 122 should not be dependent upon such an algorism for so simple an exercise with fractions. It would seem instead that he should readily recognize $\frac{1}{3}$ of 3 fourths as 1 fourth. The point to be emphasized here, however, is that the incorrect answer of 4 did not reveal this boy's dependence upon an algorism as did the interview.

It helps to make such interviews more effective to plan for them by preparing exercises which pupils are asked to work with and which will help bring out points desired. For example, it helps to reveal a child's dependence upon mechanical manipulations in arithmetic to confront him with an exercise arranged in a manner to which he is not accustomed.

Some teachers have experimented with wire or tape recordings of interviews. They report considerable help for the pupil interviewed when he is able to hear his own responses to questions asked by the interviewer. It is indeed a rewarding experience to see a child when he hears his own expression of his faulty thinking and clearly sees his difficulty. This seems to be effective diagnostic and remedial teaching in one operation.

Other teachers report that they successfully use recorded interviews with an entire class. Care is taken, of course, that the record is one of an interview with a pupil not a member of the class and whose identity is unknown to the members of the class. At intervals when the recording is being played for the class, it may be stopped for a discussion of the answers made by the interviewee. Many pupils will see clearly the faulty practices of these answers and be highly pleased with their analysis.

Whether he makes recordings or not it is surprising how soon a teacher can make the rounds of the class if he saves out two or three papers from each set he returns to his students and uses them for individual consultations. Moreover, the time will be just as profitably spent as was that which he devoted to checking the entire set of papers.

Edited by John A. Brown, Wisconsin High School, University of Wisconsin, Madison 6, Wisconsin, and Houston T. Karnes, Louisiana State University, Baton Rouge 3, Louisiana

The status of mathematics in my school

Contributed by John M. Hollowell, Jacksonville High School, Jacksonville, Illinois

EVERY STUDENT in the four-year senior high school at Jacksonville, Illinois, is enrolled in a minimum of one year of mathematics. At any given time, about ten out of seventeen students are enrolled in some class section in mathematics. About 500 of the 840 student body take a mathematics course in any one year. This is about 60%.

Class size averages about 28 pupils. About 200 freshmen will be included in the seven sections of ninth-grade general mathematics. About 75 freshmen and 65 upperclassmen will be included in the five sections of Algebra 1 and 2. Nearly 90 students—about 65 to 70 tenth graders—will be included in three sections of plane geometry. Two sections of Algebra 3 and 4 will include 50 to 60 students from the 11th and 12th grades. A single section of solid geometry and trigonometry completes the program.

A student may meet graduation requirements in any one of three "courses" called "General," "Vocational," and "College Entrance."

In the General and Vocational courses, the requirement is at least one year of mathematics, and either two years of mathematics or two years of science. The two years of mathematics may be (a) general mathematics and algebra, or (b) algebra and geometry. The student may elect more mathematics, and some do.

In the College Entrance course, the requirement is at least one year of algebra and one year of geometry. General mathematics, if taken, is counted only as an elective in this course.

The 1953–54 school year will be the first in which a full second year of algebra will be offered, and the first year in which the student (usually college-bound) will be able to have four units of mathematics *not* including general mathematics.

There are eighteen class sections assigned to teachers as follows:

"A"—1 general math; 1 solid geometry and trigonometry
"B"—4 algebra 1 and 2; 1 algebra 3 and 4
"C"—2 general mathematics; 3 plane geometry
"D"—3 general mathematics; 1 algebra 3 and 4
"E"—1 algebra 1 and 2
"F"—1 general mathematics

Teachers A, B, C, and D are "the Math Department." All have master's degrees, two of them in mathematics, and all have taught in the school twenty-three years or more.

These teachers believe our school should have "double-track" offerings in mathematics to better meet the needs of students who will not go on to college. It would also make possible having higher standards in the college preparatory sections. This has been recommended for several years, but so far the administration has not approved the recommendation.

Some other factors involved in the situation are:

More than one-third of the student body comes from rural areas.

There is very little industry in Jacksonville, although industry is growing in this area.

Two small liberal arts colleges are located in Jacksonville. One is a girls' school.

A high-school teacher looks at arithmetic

Contributed by Phil Clamurro, Arts High School, Newark, New Jersey

What's the matter with elementary school teachers? If they'd teach these kids some arithmetic, we could do a job with them in algebra! Many of us teaching mathematics on the secondary level have thought and said these things many times. But, don't be alarmed and hurry to defend yourselves, my elementary friends. We get it too, you know—from the colleges. We, too, do a "poor" job teaching high-school mathematics! That's why our college instructor colleagues can't do a real job there!

Let me say right now that there will always be the need for the high-school teacher to teach arithmetic at every level —to fill in gaps and deficiencies in arithmetic whenever and wherever we find them. And it is our obligation to do just that. The present Newark course of study, for example, very consciously puts much time and great emphasis on arithmetic review in first year algebra. This is a very good approach and is in line with the thinking expressed here. Let the colleges do likewise wherever they encounter gaps in the secondary math training of their students. This is the constructive approach—the one that holds out for the welfare of the children we teach. Otherwise, why do we hold our jobs?

Let us not, however, be complacent and accept all our problems without some thinking and evaluation. Education, indeed mathematics teaching, cannot be static. If it is, we travel backward. We can improve our mathematics instruction at every level (perhaps, indeed, at the college level most of all). We teachers at every level have the obligation to contribute to the constant improvement of instruction at our own level, and I believe we can and should contribute substantial thinking to the improvement of mathematics teaching on the other levels.

It is with this conviction that I presume to set down here a few comments on arithmetic from the viewpoint of a high-school teacher. First, I believe that all along the line on the educational front, and in arithmetic teaching too, we should do fewer things and do them well. Let's halt the headlong rushing to do many things in the elementary and secondary schools which are now, and should be, common schools for all the children.

Secondly, I believe we must spend much more time in arithmetic on developing and fixing the few concepts involved; securing number insights; and having children get the "feel" of numbers. (Again, this same comment can be applied with as much force to ourselves as high school teachers.) We can and must teach with meaning in arithmetic, and it is essential for the young learner to get these insights at every level if there is to be any real retention of skill.

There are but two basic needs in arithmetic: two basic questions to be answered —how many, and how much? *How many* involves the concept of counting, and *how much* the concept of measurement. To accomplish a better and more widespread grasp of these insights, we can make more use of the number ladder (counting) technique of teaching the addition and sub-

traction combinations for example. A greater emphasis on ten and its multiples should also be beneficial at this level.

The concept of measurement needs more stress in multiplication and division. This should be of particular benefit in the area of fractions. One of the heartaches for high-school teachers is: $4 \div \frac{1}{2} = 2$? Let's mark off (measure) $\frac{1}{2}$ along our number ladder to get the proper answer, and let's spend a lot of time with it. Let's take more time to get to the abstractions and the rules in arithmetic.

My third suggestion would follow naturally from my second. More time spent developing insights will leave less time for drill, which is all right with me. Drill with a real purpose is the only worth-while drill. Better problem selection and even topic selection with less long column addition; very large number multiplication and division; fractions with denominators other than 2, 3, 4, 5, 6, 8, 10, 12 may well be soft-pedaled or even deleted.

I looked into my son's notebook recently (fourth grade), and noticed among his arithmetic notes a list of the Roman numerals up to 50. Let's ask ourselves "why," and let's also encourage our pupils to ask "why." Only by a process of intelligent selection and rejection can we find enough time to do anything well!

We should even save enough time from drill (or it may even be that we must argue for more time for arithmetic teaching in the daily schedule) to spend on the solving of word problems in arithmetic. This is my fourth point. This is a sore spot in arithmetic and a big stumbling block in high-school algebra. I believe we can get some real help here from our elementary teacher friends. I have an idea that we don't give it enough time, that we rush young people's thinking too fast in problem solving and attempt some problems too early in the high school. Can't we have more problem solving with a real design in the grades?

My fifth comment is that algebraic concepts can and should contribute insights toward better arithmetic understanding. Thus, just one example is the concept of combining similar terms only—we may add or subtract numbers only in the same category and get an answer in that same category.

Thus, remember $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$? How can any of us forget this type of error! "Similar terms" says that they can't be combined until we make them similar: $\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$. And why must we line up the decimal point in addition and subtraction? Because the teacher says so! "Similar terms" says so. A better articulation of high school mathematics teachers and elementary teachers, an easier liaison, comparison of notes, and discussing common problems can be of much help here.

The other day I asked my class in algebra why we "indent" in the one example given below and not in the other. You can guess the answers I got; none had real insight.

$$\begin{array}{r} 20 \\ \times 5\frac{1}{4} \\ \hline 5 \\ 100 \\ \hline 105 \end{array} \qquad\qquad \begin{array}{r} 56 \\ \times 32 \\ \hline 112 \\ 168 \\ \hline 1792 \end{array}$$

Algebraic methodology and the distributive law of multiplication naturally explain this one. $56(30+2)1680+112=1792$. Perhaps we ought to un-streamline this algorithm and do it this way all the time:

$$\begin{array}{r} 56 \\ \times 32 \\ \hline 112 \\ 1680 \\ \hline 1792 \end{array}$$

You see, I was also teaching arithmetic in the algebra class.

My sixth point is a brief plea for the use of more visual materials in arithmetic.

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• RESEARCH IN MATHEMATICS EDUCATION

Edited by Kenneth E. Brown, Department of Health, Education, and Welfare, Washington, D.C.

QUESTION: What mathematics should be offered in the community college? The community college is defined as the two-year college which provides both general education and occupational training.

STUDY: R. P. Bentz, *Critical Mathematics Requirements for the Program of the Community College*, George Peabody College for Teachers, Nashville, Tennessee.—Major Faculty Adviser, F. L. Wren.

More training in statistics and the use of the slide rule is needed in general and vocational education by students in the first two years of college. This conclusion is indicated by a research study by Mr. Bentz. The purpose of his study was to establish criteria for determining the content of the mathematics courses in a community college. The emphasis was on functional mathematics for the post high-school pupil. The important concepts for both general education and occupational training were considered with the emphasis on those of occupational importance.

The procedure used in the study was the "critical incident technique." Essentially the procedure consisted of the collection of descriptions of situations in which mathematics was used or needed. These incident descriptions were secured by questionnaires, personal interviews, and observations.

The areas of mathematics *most frequently* found in the situations reported in this study were:

1. *Fundamental Processes of Arithmetic*—Skill in computing accurately with integers, common fractions, decimals, and percentages.
2. *Formula*—Ability to make a proper selection of a formula from a manual or from memory, and to use it efficiently.
3. *Problem Solving*—Ability to make a selection of the significant facts in a given problem, and to apply the necessary techniques to bring about a satisfactory solution.
4. *Mechanical Calculating Aids*—Ability to use the slide rule and calculating machines to perform various fundamental operations.
5. *Analysis of Data*—Ability to read intelligently statistical generalizations, and to construct a frequency table and statistical graph.

There were 237 usable incidents reported. A more complete analysis of these incidents revealed the use or need of the following 31 mathematical concepts:

1. Skill in computing with integers, common fractions, and decimal fractions.
2. Ability to make mental calculations with reasonable speed and accuracy.
3. Familiarity with terms used in the identification of various numbers.
4. Knowledge of the principal units that are to be found in common usage.
5. Skill in changing from one set of units to another.
6. Understanding of the nature of and the ability to use the techniques of percentage.
7. Ability to interpret and express a relationship by means of a chart, formula, or graph.
8. Skill in setting up and solving simple equations to find the value of an unknown number.
9. Ability to make proper selection and use of a formula from memory or from a reference source.
10. Skill in setting up and making use of a proportion.
11. Ability to carry out an interpolation.
12. Ability to construct and interpret bar, circle, and line graphs.
13. Understanding of the usefulness of a system of co-ordinates.
14. Understanding of the meaning of the more common symbols used in the field of mathematics.
15. Knowledge of the fundamental properties of the common figures as the square, rectangle, circle, triangle, rectangular solid, sphere, cylinder, cone, and cube.
16. Ability to make and use a scale drawing.
17. Ability to make use of the 3-4-3 right-triangle relationship and to apply the rule of Pythagoras.
18. Skill in making approximations of distances, areas, and volumes without the aid of the more precise measuring instruments.

19. Skill in making direct measurements with the protractor, scale, tape, micrometer, and other instruments.
20. Ability to collect and to tabulate accurately various kinds of numerical data.
21. Understanding of the significances of such fundamental statistical measures as the arithmetic mean, median, mode, range, and standard deviation.
22. Ability to compute an average.
23. Ability to read intelligently the statistical generalizations in various types of publications.
24. Ability to construct a frequency table and a statistical graph.
25. Ability to use the slide rule and calculating machines to perform various fundamental calculations.
26. Understanding of the meaning of a logarithm and the ability to use it as a short cut in making calculations.
27. Skill in the use of the sine, cosine, and tangent ratios in determining distances and angles.
28. Awareness of the importance of doing careful, accurate work and of checking results.
29. Awareness of the importance of developing correct habits of a clerical nature in writing figures, organizing work, and identifying results.
30. Ability to deal intelligently with the matters of loans, investments, and the cost of borrowing money.
31. Ability to make a selection of the significant facts in a given problem, and to apply the necessary techniques to bring about a satisfactory solution.

These concepts, with the exception of mechanical calculation and statistics, are usually included in the offerings of the secondary school. Mr. Bentz points out that the list of critical requirements represents the "minimum amount of mathematics" for persons entering the fields of business, industry, and homemaking. Perhaps it should be pointed out that the identification of situations requiring mathematics was made by persons, many of whom had limited training in mathematics. Without an understanding of mathematics it is difficult, if not impossible, to see the mathematical implications of a life situation. If the individuals giving the incidents in which mathematics was used or needed had possessed greater mathematics knowledge, they might have seen uses for other mathematics concepts. Mr. Bentz says "a person should have more training in mathematics than that which is indicated by this list of basic needs." Included in this training Mr. Bentz suggests that more emphasis be given to instruction in statistics and in the use of the slide rule. The frequency of occurrence of incidents involving the slide rule and other mechanical calculators indicates that training in this area is very important for the persons in this study.

Mr. Bentz states that the importance of statistics to industry, business, and homemaking is on the increase. He suggests that the statistical training should be provided that will show (1) the meaning of statistical terms, (2) the use of statistical methods, and (3) the advantages and limitations of the methods.

"Mathematics is not the dry and deadly thing that our teaching of it and the uses we put it to have made it seem. Mathematics is the hand-writing on the human consciousness of the very Spirit of Life itself. Others before Pythagoras discovered this, and it is the discovery which awaits us too."—Claude Bragdon.

". . . we love to discover in the cosmos the geometrical forms that exist in the depths of our consciousness. The exactitude of the proportions of our monuments and the precision of our machines express a fundamental character of our mind. Geometry does not exist in the earthly world. It has originated in ourselves. The methods of nature are never so precise as those of man. We do not find in the universe the clearness and accuracy of our thought. We attempt, therefore, to abstract from the complexity certain relations susceptible of being described mathematically."—Alexis Carrel, *Man the Unknown*, p. 8.

(*Answer to puzzle on page 31.*)

[1] 1	[2] 1	[3] 0	X	[4] 6	[5] 5	X	[6] 1	[7] 6
[8] 2	2	2	[9] 2	X	[10] 4	[11] 0	X	[12] 0
X	[13] 5	7	X	X	X	[14] 0	[15] 0	0
X	X	X	X	X	[16] 3	0	1	X
X	[17] 1	[18] 0	[19] 0	[20] 4	1	8	0	X
X	[21] 2	8	1	X	X	X	X	X
[22] 2	5	0	X	X	X	[23] 8	[24] 4	X
[25] 8	X	[26] 1	[27] 1	X	[28] 1	8	6	[29] 4
[30] 4	[31] 0	X	[32] 2	[33] 1	X	[34] 1	0	9

What's new?

BOOKS

Secondary

Algebra for Problem Solving Book 2, Julius Freilich, Simon L. Berman, and Elsie Parker Johnson, Boston, Houghton, Mifflin Company, 1953. Cloth, i+511 pp., $3.20.

College

A Brief Survey of Modern Algebra, Garrett Birkhoff and Saunders MacLane, New York, Macmillan Co., 1953. Cloth, v+276 pp., $4.75.

Coordinate Geometry, C. O. Tuckey and W. Armistead, New York, Longmans, Green & Co., Inc., 1953. Cloth, ix+464 pp., $2.50.

An Introduction to the History of Mathematics, Howard Eves, New York, Rinehart & Co., Inc., 1953. Cloth, vii+422 pp., $6.

Mechanics, Keith R. Symon, Cambridge, Addison-Wesley Publishing Company, Inc., 1953. Cloth, vii+358 pp., $7.50.

Plane Trigonometry with four-place tables, Arthur W. Weeks and H. Gray Funkhouser, New York, D. Van Nostrand Co., Inc., 1953. Cloth, iii+197 pp., $2.88.

Sample Survey Methods and Theory, Volume I, Morris H. Hansen, William N. Hurwitz, and William G. Madow, New York, John Wiley & Sons, Inc., 1953. Cloth, vii+638 pp., $8.

Sample Survey Methods and Theory, Volume II, Morris H. Hansen, William N. Hurwitz, and William G. Madow, New York, John Wiley & Sons, Inc., 1953. Cloth, vii+332 pp., $7.

Miscellaneous

Engineering Statistics and Quality Control, Irving W. Burr, New York, McGraw-Hill Book Co., Inc., 1953. Cloth, vii+442 pp., $7.

Mathematics for All High School Youth (report of basic skills conference-clinics in mathematics), New York State Education Department, Albany, N. Y., 1953. Paper, 2+108 pp., 50¢.

Pasteur's and Tyndall's Study of Spontaneous Generation, James Bryant Conant, Cambridge, Harvard University Press, 1953. Paper, 2+61 pp., $1.25.

Problems for Descriptive Geometry—a pictorial approach, Harold Bartlett Howe, New York, The Ronald Press Co., 1953. Paper, iii+77 pp., $3.50.

PAMPHLETS

Blueprint for Tomorrow
Institute of Life Insurance
288 Madison Avenue, New York 22, N. Y.
Insurance pamphlet, 1953. Free.

A Guide for Instruction in Mathematics
Syndicate Press
501 Park Avenue, Minneapolis, Minn.
Curriculum Guide, 1953. $1.

FILMS

A Day Without Numbers
Audio-Visual Materials Consultation Bureau
Wayne University, Detroit, Mich.
Film, 1953. Color $75; B&W $40.

Piercing the Unknown
International Business Machines Co.
590 Madison Avenue, New York 22, N. Y
Film, 1953. Color, Free.

INSTRUMENTS, DEVICES, AND MODELS

Stadia Tube
Elwood M. Stoddard
47A Jones Street, Hingham, Mass.
Field Instrument, 1951. 50¢.

The Fraction Trainer
Robinson-Howell Company
641 Mission Street, San Francisco 5, Calif.
Teaching Device, 1953. $3.75.

Numbermaster
Robinson-Howell Company
641 Mission Street, San Francisco 5, Calif.
Teaching Device, 1953. $32.50.

Theory of Flight Kit
Models of Industry, Inc.
2804 10th Street, Berkeley, Calif.
Model, 1953. $52.

PRICE LIST OF REPRINTS

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National Science Foundation

Leon W. Cohen, new program director for the Mathematical Sciences at the National Science Foundation, is offering his assistance in making the support for research offered by the foundation available to the mathematical community.

It is expected that the bulk of the grants for the Mathematical Sciences will be made in March, 1954, for activation then or in the summer or fall. It is desirable that proposals for such grants reach this office by December 31. The Advisory Panel for the Mathematical Sciences will probably meet in February, 1954. It consists of:

R. P. Boas, Northwestern University
Salomon Bochner, Princeton University
W. E. Duren, Tulane University
K. O. Friedrichs, New York University
H. Hotelling, University of North Carolina
D. H. Lehmer, University of California (Berkeley)
Saunders MacLane, University of Chicago
E. Reissner, Mass. Institute of Technology
Hassler Whitney, Institute for Advanced Study, Chairman

A guide for the preparation for proposals is available and will be sent upon request. All correspondence should be addressed to:

Leon W. Cohen
Program Director for Mathematical Sciences
National Science Foundation
Washington 25, D.C.

Dates to remember

April 21–24, 1954. Thirty-second annual convention at Cincinnati.

What is going on in your school?

Continued from page 50

Many good ones have been developed, but too often they seem to remain the particular possession of teachers and college instructors giving demonstrations at teacher conventions. Many can be made simply by the teachers themselves if they could only find the time. In these days of tremendously large classes in the elementary school, I don't want to wish any more work upon the teacher. I'm reminded of a board-member friend in a small town who told me of the large enrollment in his school and, in particular, I'm reminded of the fourth-grade teacher, who had forty-eight pupils. "Why don't you hire another teacher?" I asked. "That would make two reasonably sized classes."

Reply: "She's a strong teacher and can handle the situation. We'll hire another teacher next year when the kids get to the fifth grade."

I wonder how much real arithmetic these kids can get even with this "strong" teacher.

Finally, I would recommend constant stress in arithmetic on estimating results beforehand and checking results afterward. The longer I teach, the more convinced I become of the great worth of these techniques of every level of mathematics instruction. I believe it can be especially helpful in percentage, work with simple formulas, and verbal problems generally.

A final word to my high-school colleagues is in order. After our elementary teacher friends have taken my counsel and the advice of others more qualified than I on problems in the teaching of arithmetic—remember—you will still have to do a better job than you now do on arithmetic in your mathematics courses in grades 9, 10, 11, and 12! If we on both levels do this better job, then we can all feel that we've discharged our obligation to these children. College instructors, you may carry the ball from this point!—*From The New Jersey Mathematics Teacher.*

Reviews and evaluations

Edited by Cecil B. Read, University of Wichita, Wichita, Kansas

College Algebra, Jack R. Britton and L. Clifton Snively, New York, Rinehart & Co., Inc., 1953. Cloth, x+502 pp. $5.

Those who have used the authors' *Algebra for College Students* will welcome their new book. It is a revision of the older book in which the elementary material through Quadratic Equations has been reduced from eleven chapters and 219 pages to nine chapters and 169 pages. The chapter on advanced topics in quadratic equations has been incorporated in the first chapter on the same subject, and the chapter on "Theory of Equations" has been expanded to include the solution of the general cubic and quartic equations. Two new chapters have been added, one on partial fractions and the other on finite differences.

The book has the same attractive appearance, clear explanations, ample supply of illustrations and exercises as the older book. There is a table of powers, roots, and reciprocals, a table of five place common logarithms, and a table of natural logarithms. Answers to the odd-numbered problems are given. The book is most heartily recommended for the college freshman who has some high-school preparation in algebra.—Frank C. Gentry, University of New Mexico, Albuquerque, New Mexico.

Differential and Integral Calculus, Philip Franklin, New York, McGraw-Hill Book Co., Inc., 1953. Cloth, xi+641 pp., $6.

This textbook gives a very thorough covering of introductory differential and integral calculus. In fact it seems doubtful that any instructor would find any material appropriate to a first calculus course that is not included. A possible exception might be the answers to the exercises—they are not included.

The text contains about forty-six pages of review material in trigonometry and analytic geometry. These paragraphs are marked with an *R* and are printed in smaller type (probably an 8-point type with the regular material printed in a 10-point type).

Many paragraphs are marked with asterisks. They supposedly contain material "for the benefit of the thoughtful student and the student who finds it worth while to give the work a second reading to deepen his grasp of the subject." There are about 117 pages of this material in smaller type.

The review material, material marked with an asterisk, illustrative examples, and exercises are all printed in the smaller type. Since this material represents almost half of the textbook, it means a much more extensive book than the 626 pages of text and exercises would appear to represent.

The book also contains a short list of thirty-four integration formulas which would appear to need supplementing by one of tables of integrals for the usual course of integration. Four-place tables of natural and common logarithms are included. Also there is included a page table of exponential and hyperbolic functions.

There are several points that might disappoint a calculus instructor. For example, pages 7 to 21 cover the basic ideas of limit, function, infinity, operations with infinity, continuity, and infinitesimals in paragraphs labeled with asterisks, which could give a student a false impression of its need and value. Indeterminate forms are taken up very late in the text (page 452) along with Taylor's Series. Volumes of revolution are first considered on page 359 and could be improved by better figures. Centroids of areas are taken up on page 370, adequately covered, but the misprint on page 371, equation (27), seems quite unfortunate as many students have difficulty in clearly understanding this material.

Many paragraphs are labeled with asterisks that an instructor would want to include if possible in an eight-hour calculus course, such as "Limit of a Variable," "Operation on Limits," "Infinity," "Roots by Newton's Method," "Derivative of log *n*" (base 10), "Differential of Arc Length" (Polar Coordinates), "Improper Integrals," "Taylor's Series With a Remainder," "Maclaurin's Series With a Remainder," and possibly others. On the other hand, there are several paragraphs which are not labeled with asterisks that may not be included in every calculus course, such as "Motion in a Curve" (parametric form), "Simple Harmonic Motion," "Work Done by an Expanding Gas," the chapter on "Vectors and Surfaces in Space" (at least the part involving Vector Analysis), and the chapter on "Differential Equations."

Especially to be commended are the style, rigor, and completeness of this text. The author has done a very fine job of maintaining accuracy and rigor with clarity and reasonable brevity in his introduction of all topics. The illustrative examples with each section are well chosen and neatly worked out. Each section is followed by about twenty well-graded exercises. This text should prove very adaptable to any calculus course.—Charles B. Tucker, Kansas State Teachers College, Emporia, Kansas.

Elementary Mathematics From An Advanced Standpoint, Volume I: Arithmetic, Algebra, Analysis, Felix Klein, New York, Dover Publications, Inc. (trans. from the 1924 edition). Paper, ix+274 pp., $1.50.

This is a translation from the original German by the very capable translators E. R. Hedrick and C. A. Noble. The original style of writing of Klein has been preserved, making the book very readable since Klein presented mathematics to his students not as "isolated disciplines" but as an "integrated living organism." This book contains the background material with which Klein as a master teacher felt prospective teachers of mathematics should be acquainted. It is the first of three volumes in which other portions of the field of mathematics are covered in a similar concise but thorough manner.

Topics under the heading of "Arithmetic" include: "Calculating with Whole Numbers," "Extension of the Notation of Numbers," "Special Properties of Integers," and "Complex Numbers"; under "Algebra": "Real Unknowns," and "Equations in the Field of Complex Quantities"; under "Analysis": "Logarithmic and Exponential Functions," "Goniometric Functions," "Concerning Infinitesimal Calculus Proper," "Transcendence of the numbers 'e' and 'π'," and "Theory of Assemblages."

Each of these topics is treated from an advanced scholarly point of view. The book contains a wealth of suggested original sources where additional material on each of the topics covered may be obtained. The special index of names which is included shows that few contributors to these branches of mathematical knowledge have been neglected in this volume.

The reviewer especially recommends the book for classes in the history of mathematics and for the library of every mathematics teacher, since there is nothing comparable to it either from the standpoint of the coverage of the material it contains or from the skilful manner in which this material is presented.—HERBERT HANNON, Western Michigan College of Education, Kalamazoo, Michigan.

Essential Business Mathematics (2d ed.), Llewellyn R. Snyder, New York, McGraw-Hill Book Co., Inc., 1953. Cloth, 421 pp., $4.50.

This book provides a wealth of very practical information about the many types of computations that are so essential to the operation of an efficient business. Although the first part of the book consists of a review of simple arithmetic calculations that are so frequently used, the major part of the book introduces a very wide range of business problems on a quite advanced level as algebraic formulas are used where they provide the most efficient method of solving problems.

This realistic approach probably puts the material above the level of the average high-school student and therefore the book would be used most advantageously at the college freshman level. Business college and junior college teachers of accounting might well use this material for a semester before introducing actual accounting problems. High-school mathematics teachers could use many of the problems to supplement the material contained in the typical high-school algebra textbook as many students find that problems involving taxes, retail prices, insurance, etc. help them to grasp the principles developed in more intangible ways in some mathematics texts.

This book is therefore worthy of consideration by any teacher who is especially interested in the more practical applications of relatively simple arithmetic and algebra.—JOHN W. RAU, JR., Head of Business Dept., New Trier Township High School, Winnetka, Illinois.

Flatland—A Romance in Many Dimensions (6th ed.), Edwin A. Abbott, New York, Dover Publications, Inc., 1952. xx+103 pp., paper $1; cloth $2.25.

In his introduction to this printing, Banesh Hoffmann refers aptly to this seventy-year-old fantasy as "a timeless classic of perennial fascination" which "defies the tyrant Time." The author, once a classicist headmaster of The City of London School, speaks through the memoirs of a narrator, A Square, who is an inhabitant of Flatland, a domain peopled by plane geometric figures. From his vantage point of space, the reader, even without the benefit of formal training in mathematics, can appreciate the nature of two as well as one dimensional domains, and can sympathize with Sphere's discomfiture when an enlightened A Square argues the possibility of four or more dimensions. Flatlanders are subject to earthly restrictions and feelings: sharp angles are dangerous, for example; regularity is a virtue; and the narrator, imprisoned for preaching the heresy of 3-space, expresses the hope of ultimate enlightenment for his fellow polygons of Flatland.

The booklet, which has appeared previously in German as well as in English, is recommended as instructive, entertaining, and stimulating to the imagination.—JAMES M. EARL, University of Omaha, Omaha, Nebraska.

A History of Astronomy from Thales to Kepler J. L. E. Dreyer, New York, Dover Publications, Inc., 1953. Paper, x+438 pp., $1.95.

This is a republication, under a revised title, of *History of the Planetary Systems from Thales to Kepler*. The original work, reported out of print, has been revised by W. H. Stahl, but apparently the revision consists of little more than an added foreword and a supplementary bibliography, which includes recent references.

The book will find its greatest use as a reference volume, for the most part in college libraries. Many readers will find it heavy reading, not only because of the meticulous detail with which the subject is discussed, but also because of the need for knowledge of Greek, Latin, and German in order to understand some of the material. For the reference purpose cited, there is prob-

ably nothing to equal this work.—CECIL B. READ.

Linear Algebra and Matrix Theory, Robert R. Stoll. New York, McGraw-Hill Book Co., 1952. xv+272 pp., $6.00.

This book is a text designed to introduce the concepts of modern algebra to advanced undergraduates or first year graduate students via vectors, linear transformations, quadratic forms and matrix theory. The concepts of abstract field, ring and group are introduced and discussed when they arise naturally. Determinants are treated axiomatically. It is a solid, well-written exposition of the basic material on these subjects, the best of its type that the reviewer has seen. Topics treated, in addition to those already mentioned, include linear equations, canonical representation of linear transformations, and unitary spaces.—WILLIAM BOOTHBY, Northwestern University, Evanston, Illinois.

The Methods of Statistics (4th ed.), L. H. C. Tippett, New York, John Wiley & Sons, Inc., 1952. Cloth, 395 pp., $6.

This book is well ordered, covers the field of statistics in a systematic manner, and explains well the basis of practically all methods of statistics in use today. There are many well-chosen examples to illustrate the techniques to be used, and many tables and charts are provided to make the methodology workable and practical. A good system of notation has been adopted. Mathematical proofs, when not too difficult and when likely to aid understanding have been given. The reader is introduced quite early to moments and computation of moments, from both ungrouped and grouped data. With mathematical and statistical probability as a foundation, the Binomial, Poisson, Exponential, and Normal Distributions, their uses and limitations, their associated tables and charts, are also presented quite early. Considerable attention is devoted to the theory of sampling, both large and small samples, and particular attention is called to the reliability of the conclusions reached through sampling, or sampling errors. Various applications of the analysis of variance are given special emphasis, more fully dealt with than anything else. A working knowledge of elementary calculus is necessary for the reading of this book. A thorough understanding of the earlier chapters of the book is necessary for the understanding of the later ones. The reader will find guidance in a very extensive list of references if he desires to continue some branch of the subject beyond the point to which the author has taken him. The book, *Technological Applications of Statistics*, by the same author (1950) is strongly recommended as supplementary reading.—FRANK C. GERMAN, Kansas State Teachers College, Pittsburg, Kansas.

Modern Elementary Statistics, John E. Freund, New York, Prentice-Hall, Inc., 1952. Cloth. x+418 pp., $5.50.

This book covers the essential content of basic statistical material, more than probably would be covered in a single semester. The explanations seem unusually clear, yet not verbose; this would be a book of benefit to the student having trouble with certain topics. There is a well-selected list of references at the end of each chapter. Illustrations seem very complete and supplement the text. Problems are given at the end of the chapter, although some instructors might believe that the list is too short to avoid duplication over several semesters. Tables and answers to odd-numbered problems are at the back. There is no supplementary material available. Typography and arrangement of material is very good. This book will make a good source of supplementary material and some instructors may wish to give it serious consideration as a class text.—ROBERT H. WATSON, University of Wichita, Wichita, Kansas.

Plane Geometry, Virgil S. Mallory and Chauncey W. Oakley, Chicago, Benj. H. Sanborn & Co., 1953. Cloth, viii+468 pp., $2.48.

This is a text that will appeal to both student and teacher. When a teacher of mathematics looks for a good text, he looks for a *teaching aid*, not an instrument that aims to *do* the teaching. The teaching is his job. However, he does hope to find a text that is written by someone who knows what should be taught and how. Here is a text that should satisfy the requirements of a good teacher of plane geometry. The authors themselves are teachers of the subject. Their combined experiences in teaching plane geometry have made it possible for them to produce an excellent text. It is written for the student in language he can understand—clear, accurate, interesting. Stress is on thinking. Teaching helps are adequate.

The text has two objectives, the training of future scientists and teaching students to think intelligently. These two important objectives are well carried out in the content and its arrangement. In fact, it is the opinion of the reviewer, a teacher of geometry for many years, that the claims made for the text by the authors in their preface are actually justified in the text itself. Certainly, this text provides ample help for a good teacher to realize these two very important objectives.

Some of the high spots of the text are: gradual introduction to proof, emphasis on thinking, easy transition from algebra to geometry, several of the more difficult theorems postulated at first and proved later, emphasis on algebra in the treatment of ratio and proportion, a excellent optional chapter on numerical trigonometry, optional material on related three-dimensional geometry in which proofs are not required, practical applications emphasized throughout by the use of well-selected problems and pictures, well-selected historical notes, a very good treatment of locus with an easy concept being given early in the course and treated specifically in Chapters VII and XIII, ample provision for reviews, summaries of concepts and skills at the end of each chapter, "Keyed Practice" on each chapter followed by a "Final

Test," cumulative tests at the end of each chapter after the first.

Problems are well selected and are grouped in three parts: for a basic course only, for a more extensive course, and for a course aiming to meet the needs of the most interested and most capable students. The text is highly suitable and adequate for a class of selected high-ability students, as well as for an unselected group. The indicated minimum course meets all standard requirements. The basic course is arranged in fourteen chapters with flexibility for selection of materials.—H. W. CHARLESWORTH, East High School, Denver, Colorado.

Plane and Spherical Trigonometry with Tables (second ed.), Donald H. Ballou and Frederick H. Steen. Boston, Ginn and Company, 1953. vi+302 pp., $3.50.

Plane Trigonometry with Tables (second ed.), Donald H. Ballou and Frederick H. Steen. Boston, Ginn and Company, 1953. vi+244 pp., $3.25.

These are carefully written texts with excellent format and adequate coverage of all topics usually included in college trigonometry courses. A chapter on graphs includes the general sine curve and an introduction to simple harmonic motion. Chapters on complex numbers and logarithms are included. The text listed first in the foregoing includes three chapters on spherical trigonometry which contain the usual formulas and methods for solving spherical triangles; haversine formulas are applied to problems of great-circle sailing and celestial navigation. The exercises in these texts include many interesting practical problems with a modern flavor. Indicative of the quality of these texts are the treatment of the trigonometric functions for quadrantal angles and the treatment of identities. They are rigorous, yet not too sophisticated or wordy. —LAWRENCE A. RINGENBERG, Eastern Illinois State College, Charleston, Illinois.

Practical Mathematics (fourth ed.), Claude I. Palmer and Samuel F. Bibb. New York, McGraw-Hill Book Company, 1952. Cloth, 769+7 pp., $4.50.

The content of this book represents an attempt to present in one text the essentials of arithmetic, algebra, geometry, and trigonometry, with practical applications. The book is designed for use in adult-education and college-refresher courses, and for home study.

The outstanding features of the book are (1) the large number and variety of practical applications, (2) the informal development of geometry, (3) the many practical tables and formulas, and (4) the answers to all examples and problems are given.

Some weaknesses of the book are (1) the lack of development of the understanding of basic principles, (2) too much reliance upon formal definitions and rules, (3) no presentation of the fundamental operations with whole numbers, (4) the use of unnecessarily complex and confusing computational forms, (5) the use of meaningless examples, such as "What is the difference between the sum of 2/21 and 3/35 and the product of 3/7 and 28/183?" and (6) a poor choice of vocabulary, and long complex statements.

This book provides an excellent source of supplementary problems for high school mathematics classes. Mechanics and sheetmetal workers should find many valuable aids in this text, provided that they already have a facility with arithmetic.—DALE CARPENTER, Los Angeles City Schools, Los Angeles, California.

The Rational and the Superrational (Studies in Thinking), Volume II. Cassius Jackson Keyser. New York, Scripta Mathematica, Yeshiva University, 1952. Cloth, viii+259 pp., $4.25.

"The essays in this volume are offered to the public as a portrait of Keyser drawn by himself" —so writes the editor, Professor Ginsburg, in the Foreword. He could not have written a more fitting sentence. Fortunate enough to have been numbered among Keyser's students, your reviewer literally relived the poignant delights he experienced when first he thrilled to these essays. For here is Keyser speaking right from the pages before us—the expressive hands, the keen eyes, the characteristic tilt of the head, the dramatic pauses—all come to mind with amazing fidelity.

The editors are to be congratulated for having brought together material which, in the main ,belongs together by virtue of the *leitmotif*—exploring the nature of knowledge and of thinking, testifying to man's insatiable need for idealizing. Included in the collection are: "Science and Religion," "The New Infinite and the Old Theology," "Thinking About Thinking," and "Mole Philosophy and other Essays," as well as two additional essays reprinted from *Scripta Mathematica.* All the essays are as fresh and vibrant as when originally written, even though a quarter of a century separates the latest from the earliest.

Since to portray such essays faithfully in a few words is clearly impossible, we are reluctantly compelled to characterize them somewhat as follows. The first is at once austere and humble: "Science is the good life. . . . The mystery of knowledge and understanding is more awe-inspiring than that of the dark unknown." The second essay is both brilliant and audacious: "We transcend the incompatible by taking *all* of many things when any is allowed." The third essay is indeeded incisive: "Postulational thinking is highly important . . . and the method is available for all fields of thought. Postulate detection . . . shows us how hard it is to know; it fosters scientific modesty, discourages dogmatism, favors tolerance, and makes for the maintenance and advancement of good will in the world."

Among the miscellaneous short essays, the following gems can be commended: "Mathematics and Man," (p. 237 ff.); "Hard-to-Ask

Questions," (p. 221 ff.); "A Unique Test of an Educated Person," (p. 216); and "Self-Portraits as Drawn by Book-Reviewers," (p. 220). In short, this is a splendid addition to the first volume of the collected works of an accomplished mathematician, a distinguished scholar, and a great teacher.—W. L. SCHAAF, Brooklyn College, Brooklyn, N. Y.

Science in Daily Life, Francis D. Curtis and George Greisen Mallinson, Boston, Ginn & Co., 1953. Cloth, xii+570 pp., $3.96.

With the increases in secondary school enrollments that are expected during the next several years, *Science in Daily Life* could very well fit the teaching needs of the science instructor who will be confronted with large classes of high school pupils for whom a general type of education must be planned.

As most teachers know, the modern text is a course of study as well as a source of information. Considering the encyclopedic nature of present-day general science instruction, this well-planned, comprehensive text covers a wide range of information—all of which has place in cultural and knowledge background of today's youth.

In fact, the beginning science teacher will find it invaluable in course organization, work assignments, study aids, and illustrative materials. For the brightest children, a large measure of self-instruction, through reading, is possible.

The experienced instructor will take its wealth of material, delete those items of less essential nature or assign them for general reading and then develop in detail those science fundamentals which have carry-over value in many areas of scientific knowledge.

The subject matter, and particularly the illustrative materials, are geared to the world of young people and within their experiences in the world about them. The print is large and no effort has been spared in making the book attractive.

From the more critical point of view, the development and presentation in this text, and of others recently published, raises the question: What is general science—an information course or a background for future study of the more specialized sciences? It appears now that our text writers and science education specialists have concluded that the informational aspects of the instruction are the most important.

Then, too, just how far can we go in the organization of any text as a teaching tool, easing the job for the teacher, and making it as simple as possible for the pupil? The trend now evident indicates that it will take considerable skill on the part of the teacher to keep the pupils (and perhaps the teacher as well) from getting lost in the highly developed plan of the text organization.

Whatever the teacher's idea as to the place and purpose of General Science in the school curriculum, *Science in Daily Life* merits consideration as a teaching tool.—CARROL C. HALL, Springfield High School, Springfield, Illinois.

Solid Geometry: A Clear Thinking Approach, Leroy H. Schnell and Mildred G. Crawford. New York, McGraw-Hill Book Company, 1953. x+198 pp., $2.96.

This book is somewhat more traditional in its approach than the authors' *Plane Geometry: A Clear Thinking Approach*. However, the pedagogical features upon which the plane geometry is based have been retained.

It seems that the authors made every effort to apply geometric reasoning to everyday situations. At the ends of several chapters, everyday reasoning exercises are given in the "Keep the Iron Hot" sections. The "Keep the Iron Hot" sections also serve to review previous mathematics courses and to prepare the student for material to be presented in the next chapter.

The introductory chapter is especially well written and leads the students from their background in plane geometry to the three dimensional problems. A number of statements or sections throughout the text serve to "sell" mathematics to the students as well as to stimulate them to further study. For example, on page 119 one finds the statement, "It is hoped that the foregoing discussion will convince you that a study of analytic geometry can be fascinating as well as practical."

The drawings are clear and distinct but some pages seem rather full having more than forty lines with seventy or more characters to the line. A number of well-selected illustrative pictures appear throughout the book and each has a definite bearing on the text at the point where it is introduced.

All the required theorems listed in the National Committee Report and by the College Entrance Examination Board have been included, as well as those listed in the Syllabus of the University of the State of New York. In the main the traditional order of content is followed. The assumptions, theorems, and corollaries of plane geometry are listed in the reference chapter as well as a summary of the assumptions and formulas of solid geometry. Chapter reviews and tests are given for most of the chapters.

Teachers of solid geometry should examine this text before making an adoption.—CLYDE T. MCCORMICK, Illinois State Normal University, Normal, Illinois.

The Teaching of Secondary Mathematics, Claude H. Brown, New York, Harper & Brothers, 1953. Cloth, xi+388 pp., $4.

This book is well written in a style that makes for easy reading. The presentation is organized in three major divisions, as follows: Part I—*Nature and Role of Secondary-School Mathematics;* Part II—*Teaching Problems in Secondary-School Mathematics;* Part III—*Methods, Evaluation, and Preparation for Teaching*

Secondary-School Mathematics. While the book contains a great deal of very helpful information and presents many thought-provoking discussions of critical issues concerning curriculum content and teaching techniques, there are some major weaknesses which limit its value. The most significant of these weaknesses may be summarized as follows: (1) The bibliographies seem very inadequate. In the first place, they are very meager. Secondly, there are only thirteen distinct references, either in footnotes or in chapter-end bibliographies, which are dated 1945 or later, and only two of these are as recent as 1950. Thirdly, the author has seen fit to cite only four references (one dated 1930 and the other three dated 1935) to THE MATHEMATICS TEACHER, a rich source of materials related to the improvement of instruction in secondary mathematics. (2) The author has omitted any reference to several important committees. The most significant of these omissions are the Cooperative Committee of the American Association for the Advancement of Science, the Commission on Post War Plans of the National Council, and the committees that worked during World War II in the interest of mathematics. (3) There seems to be an undue emphasis on Part I. The distribution of pages is as follows: Part I—140; Part II—142; Part III—85. This seems to have resulted in an inadequate treatment of such important areas as intuitive, or informal, geometry; troublesome topics in second-year algebra; the stimulation and maintenance of interest; the use of audio-visual aids; and some of the more important aspects of formal geometric proof.—F. LYNWOOD WREN, George Peabody College for Teachers, Nashville, Tennessee.

Theory of Numbers, B. M. Stewart. New York, The Macmillan Company, 1952. xiii+261 pp., $5.50.

Trying to "pique student interest," Professor Stewart has written a textbook calculated to please both mathematicians and educators. Examples of things that should appeal to the latter group are the analysis of the peg-board game *solitaire* (depending only on the concept of divisibility by two), and the slide rule mod 29, an engaging and pedagogically useful toy of the sort that mathematicians belittle and educators bring to class.

Definitions, theorems, and proofs are generally sound, the sequence of topics well chosen, and the style readable. Some topics (like quadratic forms, with only one application given) are incompletely covered because the book is specifically a text and not a reference. The contents are reasonably standard, but almost no mention is made of results from analytic number theory. On the other hand, several things appear which are not usually found in such a course (construction of rationals and reals from the natural numbers; linear transformations).

Fault can be found in a few minor matters, like using forms of induction different from the one given as "the" axiom, without mention of equivalence (most authors leave a bigger gap, omitting discussion of induction altogether). On the whole, however, the book creates a favorable impression and should prove to be a worthwhile text. The author's notes as to interdependence of chapters will help in course-planning.—T. C. HOLYOKE, Northwestern University, Evanston, Illinois.

The Works of Archimedes, T. L. Heath, New York, Dover Publications, Inc., 1897 and 1912. Paper, clxxxvi+326 pp.; 51 pp., $1.95.

The publishers have combined in a single volume Heath's work, published in 1897, and "The Method of Archimedes," issued as a supplement in 1912. The book will be of greatest value as a detailed reference work for those interested in the history of mathematics. For such purpose, the availability of an inexpensive edition is appreciated; the average teacher in high school or junior college will probably find all the information he desires in one of the standard history of mathematics texts.—CECIL B. READ.

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Continued from page 45

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